An Alternative FX Option Pricing Model: A Discrete Mixture of Normal Distributions

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Abstract

With the introduction of the Black and Scholes (1973) and R. C. Merton (1973) (BSM) option pricing model, researchers and practitioners have been continually looking to amend and extend the model to improve performance. This thesis will take an alternative approach to addressing the limitations of the BSM model. Rather than append an extension to the model by relaxing one or more of the assumptions: R. C. Merton (1976), Derman and Kani (1994), Bakshi, Cao, and Chen (1997) and Dumas, Fleming, and Whaley (1998), this thesis applied a suitable transformation to the data to elicit more information relating to its distribution to ensure compliance with the BSM assumption of normality. The aim is to identify the three stages that make up the foreign exchange (FX) option price, namely:

i. Defining the constituent elements that explain the FX spot price to model the FX market behaviour.

This thesis shows that the FX market can be represented by a system of attributes: order flow, bid-ask spread and triangulation. These attributes, although transmit information unique to their own function; also operate as a system to arrive at the quoted price. These elements will be used to remodel the FX market behaviour in describing it as a stochastic price process.

ii. Represent the FX market behaviour by an appropriate stochastic price process.

The stochastic FX price process, characterised by the system of attributes describing the FX market, defines the fundamental equation, adhering to the assumption of normality, to explain the FX option pricing formula. The Kon (1984) discrete mixture of normal distributions model was utilised in describing the market systemic function to arrive at the stochastic FX price process.

iii. Applying the BSM method to the fundamental equation proposed by the thesis afforded an alternative FX option price model.

The collective affect of each attribute results in a skewed, leptokurtic distribution for the price returns. The thesis demonstrates that the constituent pricing elements are normally distributed and affect the price distribution by a proportionate shift parameter. The modified stochastic process is the basis of the fundamental equation that is applied to the BSM methodology to arrive at an alternative, modified closed form FX option pricing model.

Under the presumption that the more exact the option pricing model the more accurate the forecasting ability, the forecasting performance of each model was compared utilising risk reversal options. Forecasting the movement in the FX spot market, the precision of the modified FX option pricing formula was compared to the market leading BSM pricing model. Hence, accepting the evaluation criterion as an indicator of the senior option pricing model, the result for the alternative FX option pricing formula were very promising. The thesis clearly demonstrates that the modified option pricing model outperforms the BSM model using trend reversal indicators but is not so definite with the directional trend indicators. The encouraging initial results confirm the necessity for this research and present opportunities for further study, namely,

i. What is the true number of N normal distributions in the Kon (1984) model? This will directly impact the size of the variance shift parameter in the fundamental equations.

ii. What pricing information is contained in an option that can be used to forecast price movements? Intuitively options contain information about forward pricing, thus interpreting this information is central to any potential trading strategies.

iii. What is the appropriate trading strategy to extract forward pricing information contained in an option offering a profitable opportunity?

Dedication

”For we are God’s own handiwork, recreated in Christ Jesus, that we may do those good works which God predestined for us” Ephesians 2:10

”Give to the Lord the glory due to His name; worship the Lord in the beauty of holiness.” Psalm 29:2

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Chapter 1

Introduction

1.1 Motivation

The foreign exchange (FX) market is very large and very liquid. Trading in FX markets was averaging \$5.1 trillion per day in April 2016, increasing to \$6.6 trillion per day in April 2019; comprising of spot transactions, outright forwards, FX swaps, currency swaps, options and other derivatives, *see* the Bank for International Settlements (BIS) Triennial Central Bank Survey. As well as spot trades that made up \$1.99 trillion per day of the market in April 2019, there is a large market for derivatives, both futures and options, which accounted for \$294 million per day in April 2019. There are numerous foreign exchange options contracts that are widely traded, both for reasons of risk control and speculation.

For European options the Black and Scholes (1973) and R. C. Merton (1973) (BSM) formula would be appropriate if the returns on the underlying currency pair were independent and identically distributed (i.i.d.) normally, or at least approximated by i.i.d. normality. Although there are numerous competing theories about the evolution of exchange rates, notwithstanding these, the empirical evidence shows that returns from foreign exchange positions are not i.i.d. normal. As a consequence, the BSM formula may not be appropriate to price European currency options. Further, it is widely accepted that the price of an option may represent a forecast of future spot currency movements. If this is so, then it is self-evident that an accurate option model could lead to improved forecasts of the spot rate. In the same way that i.i.d. normality in continuous time leads to the BSM model, the different continuous time stochastic processes that have been reported in the literature lead to different pricing models for European options both for foreign exchange and other assets. That is, there is a strong linkage between appropriate return distributions, the consequent structure of the corresponding option pricing model and any implicit forecast of future movements in the underlying.

Option contracts, particularly foreign exchange, are highly traded and typically held short term. Not surprisingly, empirical studies of exchange rate movements have benefited both from the advances in computing power and in the development of models for high frequency data. Examples are the large family of Autoregressive Conditional Heteroscedasticity (ARCH) and Generalised Autoregressive Conditional Heteroscedasticity (GARCH) processes due to Engle (1982) and many others, and models that deal *ab initio* with non-normality, for example finite mixtures and scale mixtures models. Many of these econometric time series models do not lend themselves to analysis in the continuous time framework. However, the technicality can be avoided if the price of an option is thought of as the present value of the expected payoff, generally under a risk neutral distribution.

Foreign exchange option prices can be used as an indicator of the underlying directional trend of exchange rates. Garman and Kohlhagen (1983) (GK) developed a foreign exchange option pricing model based on the BSM configuration; conceding the geometric Brownian motion asset price behaviour, however based on alternative assumptions. A measure of how accurately the Garman and Kohlhagen (1983) FX option pricing model uses the embedded hedging information is an indication of how well specified the model is. In an attempt to answer this question Dunis and Lequeux (2001) examined the informational value derived from a measure of the skewness of the price distribution of a 25-delta risk reversal: an option position that consists of being short (selling) an out-of-the-money put option and being long (buying) an out-of-the-money call option with the same maturity. Risk Reversal prices are predicated on the call and put option prices derived from the Garman and Kohlhagen (1983) FX option pricing model.

Option deltas are an indication of the degree to which the option is in-the-money or out-of-the-money. The delta rises as options are increasingly in-the-money and reduce as the options move progressively out-of-the money. At-the-money options have a value of 50-delta, suggesting a 50% likelihood of either ending up in-the-money or out-of-the-money at maturity. Hence a 25-delta option is out-of-the-money with only a 25% chance of ending in-the-money at maturity. Delta is the change in the value of an option for an infinitesimal change in the exchange rate. A standard out-of-the-money currency option offered in the interbank market is a 25-delta option, which means that the price of the option would change 0.25% for a 1% change in the exchange rate. Thus a 25-delta option is an indication of the degree an option is out-of-the-money.

Dunis and Lequeux (2001) considered the amount by which a 25-delta call option is more or less expensive than a 25-delta put option with the same maturity as a buy or sell indicator of future exchange rate movements. The information contained within risk reversals is analogous to the net buying pressure of microstructure order flow in the FX spot market. Order flow is defined as the net of positive (buyer-initiated) and negative (seller-initiated) orders and seen as a measure of net buying pressure. A positive (negative) sum means a net buying (selling) pressure. Dunis and Lequeux (2001) concluded that contrary to market expectation the one-month risk reversals did not offer any embedded information that could be used profitably in directional forecasting. The same conclusions as Dunis and Lequeux (2001) were drawn by taking a comparable empirical analysis to three-month risk reversals.

Under the assumption that the unsatisfactory performance of risk reversals is attributable to the failings of the Garman and Kohlhagen (1983) modified BSM model, given that the price returns fail to comply with geometric Brownian motion which the model is based upon, this bodes the question: can a respecified FX option pricing model more accurately use the embedded information to improve the forecasting performance of the risk reversals?

1.2 Objective

The Black and Scholes (1973) and R. C. Merton (1973) (BSM) option pricing formula has become the standard pricing model adopted by the majority of market practitioners; with only a minority dissenting: Haug and Taleb (2011). The BSM breakthrough approach leads to a pricing formula using, for the most part, only observable variables. In particular the formula does not require knowledge of either the investors preferences or beliefs about expected returns on the underlying asset. Moreover, under specific posited conditions, the formula must hold to avoid the creation of arbitrage possibilities. However the derivation is dependent on unrealistic assumptions: Dremkova and Ehrhardt (2011), Henderson (2014), Janková (2018) and Krznaric (2016). The Black and Scholes (1973) and R. C. Merton (1973) solution is not valid when the asset price dynamics cannot be represented by a stochastic process with a continuous sample path. In essence, the validity of the formula depends on whether the asset price returns observe a log-normal distribution. That is, the stochastic price process is a geometric Brownian motion. This presumes that the asset price returns are normally distributed.

This thesis will focus on the assumption that the asset price follows a geometric Brownian motion with constant drift μ and constant volatility σ. Thus the distribution of prices at the end of any finite interval is log-normal and the variance rate of the return is constant. FX option pricing models disappoint by virtue of the foreign exchange price behaviour being inconsistent with the geometric Brownian motion model describing the asset price behaviour.

The aim is to identify the three sections that constitute the price of FX options to arrive at an alternative option pricing formula, namely:

i. Determine the systemic attributes of the FX market: Chapters 2 and 3.

ii. Apply these systemic attributes as the basis for arriving at the stochastic FX price processes that conforms to the assumption that the price process follows a geometric Brownian motion (GBM) through time. Thus arriving at a log-normal distribution for price returns between any two points in time: Chapters 4 and 5.

iii. Employ this stochastic FX price process to the BSM methodology to arrive at an alternative FX pricing model: Chapters 6 and 7.

1.3 Data Source and Procedure

The Refinitiv Eikon trading system, formerly Thomson-Reuters, provided the data for this study. The spot exchange rate price history comprises the best bid, mid and ask quotes and time-stamped to the nearest second: Chapter 3. The spot exchange rate and 25-delta risk reversal price history comprises the best bid, mid and ask quotes and time-stamped to the daily closing price: Chapters 5 and 7. No information as to the transaction size or trading parties is given.

Attempts to improve the performance of the Black and Scholes (1973) and R. C. Merton (1973) option pricing model have focused on enhancements to the original model by relaxing one or more of the assumptions: R. C. Merton (1976), Derman and Kani (1994), Bakshi et al. (1997) and Dumas et al. (1998). However, the enhanced models are still contingent upon the returns conforming to the normally distributed geometric Brownian motion. These models do not address the fact that asset price returns are skewed: frequency distributions which are not symmetric, and leptokurtic: a series which has a higher peak at the mean and fatter-tails than a normal distribution with the same mean and variance.

The limitations of the BSM pricing formula should not be regarded as the consequence of model misspecification. Rather the limitations are due to the shortcomings of the asset price returns not conforming to geometric Brownian motion, which the BSM model is based upon. Presuming the asset price returns non-compliance to geometric Brownian motion undermines the BSM option pricing formula, then all models based on BSM are possibly flawed.

This analysis will take an alternative approach to addressing the limitations of the BSM model. In place of modifying or appending to the BSM model, this thesis will interrogate the characteristics of the data that cause a divergence from assumed normality. The aim is not to augment the BSM model but to respecify the data so it complies with it. The thesis will address the limitations that pertain to the underlying asset by not acceding to geometric Brownian motion in respecifying the BSM formula. This infraction of the geometric Brownian motion model impairs the performance of the BSM formula. Seeking to explain the asset's distributional behaviour and how it can be modified to conform to i.i.d. normality can potentially elevate the performance of the BSM model.

Taking an empirical approach to the FX macroeconomic fundamentals and microstructure finance: Chapters 2 and 3, and a theoretical approach to FX stochastic modelling: Chapters 4 and 5, this thesis will attempt to explain the FX market price behaviour. A new FX option pricing formula will then be derived by applying the BSM methodology to the empirical and theoretical model that embraces the FX price behaviour: Chapters 6 and 7. The forecasting performance of this new FX model will then be evaluated against the market standard BSM risk reversals on the basis that the better model should result in greater forecasting accuracy.

1.4 Contribution

Foreign exchange options are used to forecast currency directional changes. A measure of how accurately the option pricing model uses the embedded hedging information is an indication of how well specified the model is. The better the specification of the FX option pricing model the more accurately it conveys the exchange rates to the FX market. The information contained can then be utilised by practitioners as a future buy or sell indicator in the foreign exchange market. Any alternative option pricing model needs to be used by market practitioners.

The thesis makes the following contribution to the literature:

i. Identifying the systemic attributes: order flow, bid-ask spread (discreteness and clustering) and triangulation, as the information transmission mechanism for the microeconomic foreign exchange market.

ii. A study of appropriate econometric models for daily returns on a number of leading currency pairs.

iii. Use of the derived stochastic model to arrive at an alternative pricing model for European option contracts for these pairs.

iv. Derivation and testing of a closed form option pricing formula that can be used by market traders to forecast future spot returns based on risk reversals derived from the modified option pricing model.

The empirical results proved promising, the modified option pricing model outperformed the BSM pricing model for trend reversing strategies, although not conclusive. This presents avenues of further research, namely:

i. Refining the process of identifying the N normal distributions of the return price.

ii. Identifying what information is embedded in the FX option that can be used to forecast the future movements of exchange rates.

iii. Develop an appropriate option based forward trading strategy rather than the currently used spot trading strategy adopted by traders.

1.5 Structure of the Thesis

The structure of the thesis consists of three parts: determining the systemic attributes of the FX market, identifying the stochastic FX price process and deriving and evaluating an alternative FX option pricing model, and is laid out as follows:

Chapter 1 is the introductory chapter. It provides motivation for the research and outlines the objectives and the main contributions made.

Chapter 2 provides an overview of the foreign exchange market and sets out the argument for the microeconomic over the macroeconomic approach of practitioners. It then presents a review of existing literature detailing the price dynamics of the foreign exchange market. The price dynamics are explained with the introduction of the information transmission mechanism. The rationale for the mechanism is discussed with respect to the information flow between practitioners and how this flow creates the market and ultimately determines the exchange rate. Chapter 3 presents the empirical work determining the composition of the bid-ask spread and how it relates to the FX market features of triangulation, discreteness and clustering in defining the constituents of the transmission mechanism to arrive at the systemic attributes of the FX spot price.

Chapter 4 presents the properties of the price processes: diffusions, jumps and a mixture of both, adopted by alternative pricing models. The literature review explores the stochastic price process of returns for the leading pricing models and evaluates their attributes. These processes are evaluated against the systemic attributes of the FX market identified in Chapter 3 to arrive at a stochastic process that captures these features and maintains the i.i.d. normality required by the BSM model. The empirical work in Chapter 5 applies the discrete mixture of normal distributions stochastic stock process to the attributes of the FX market whilst adhering to i.i.d. normality. This chapter defines a stochastic FX process to arrive at the fundamental FX option differential equation.

Chapter 6 introduces the derivation of the BSM stock option pricing model and highlights the alternative models to the BSM formula. The literature review examines the relationship between the BSM stock model and the FX option pricing model, contrasting the assumptions and methodology of each. It also presents risk reversals and outlines the function they play in forecasting exchange rate movements. Chapter 7 applies the discrete mixture of normal distributions to the BSM methodology in deriving a modified FX option pricing model. This model is then used to derive risk reversals differenced by price and not implied volatility. The derived and market defined risk reversals are then used to evaluate the modified FX option pricing model with the standard BSM model by comparing the forecasting performance of each model under the proviso that the better model should more accurately forecast the movement of exchange rates.

Chapter 8 summarises the results, provides the implications of the modified FX pricing model based on this analysis, highlights the limitations of the research and presents future research opportunities.

There are numerous appendices containing the following:

Appendix A contains the summary statistics for the data sets used throughout the thesis and details the test statistics for the empirical work contained in Chapter 3.

Appendix B details the properties of the price processes outlined in Chapter 4.

Appendix C details the mathematical properties of the stable paretian distribution probability laws outlined in Chapter 4.

Appendix D presents the lemmas underpinning the compound events model detailed in Chapter 4.

Appendix E details the discrete mixture of normal Distributions test statistics for the empirical work contained in Chapter 5.

Appendix F details the statistical analysis of skewness and kurtosis contained in Chapter 5.

Appendix G details the Black-Scholes methodology and assumptions contained in Chapter 6.

Appendix H contains the python code used to evaluate the χ^2, Standardised Range *(SR)*, attraction and resolution tests detailed in Chapter 3.

Chapter 2

The Foreign Exchange Market

This chapter introduces the notations used in the foreign exchange (FX) market from a practitioner's point of view. The purpose of this chapter is to highlight the unique nuances of the FX market and review the relevant literature. FX risk considerations have an added level of complexity not present in the equities market. In equities it is clear whether one is considering the upside or downside risk when holding a long position in stock. Conversely currencies are traded in pairs; you are long on one side of the trade whilst simultaneously short the other. In the case of a British pound (GBP) based investor who is long in US dollars (USD) and commensurately short GBP: the investor sells GBP to buy USD, the risk considerations pertain to both sides of the trade. To understand the market behaviour the trading strategies deployed by participants are surveyed, contrasting technical with fundamental analysis and appraising the relevance of macroeconomic fundamentals and microstructure finance to explain the FX market.

For British pound against the US dollar, with ISO codes GBP and USD respectively, the market standard quote of GBPUSD represents the number of USD per GBP, writing a slash ('/') between the currency ISO codes, such as GBP/USD would read GBP per USD which is incorrect. However, to aid the reader the GBP/USD notation will be adopted throughout this work but suggests it is read as USD per GBP.

2.1 The Fundamentals of Foreign Exchange

2.1.1 The Mechanics of the FX Market

The form of foreign exchange (FX) transactions is simply exchanging a certain amount of one national currency for a certain amount of another realm. Unlike other asset classes, in foreign exchange there is no natural numeraire currency and the choice of which way round currency pairs are quoted is purely market convention. Bank interdealers may state foreign exchange quotations in one of two ways:

i. The foreign currency in terms of one US dollar, in so-called European terms, for example USD/JPY.

ii. The US dollar price of a unit of a foreign currency, American terms, for example EUR/USD.

Which way round they are conveyed in the market is determined by the ISO standards. Foreign exchange quotes can be described as either *direct* or *indirect* spot rates. A direct quote is the domestic currency price of a unit of foreign currency. An indirect quote is the foreign currency price of a unit of domestic currency. Take a GBP/USD quote priced in US dollars: this spot rate, S, is a direct quote for a dollar investor and an indirect quote from a sterling viewpoint. It follows that the reciprocal spot rate, $\frac{1}{S}$, is a direct quote for the sterling investor and indirect for the dollar side. Regardless of how the price is quoted, either as S or the reciprocal $\frac{1}{S}$, it does not have a bearing on the exchange rate. For consistency the ISO notation will be advocated throughout this work.

Adopting the ISO standards for a currency (ccy) pair quoted as *ccy1ccy2*, the spot rate S_t at time t is the number of units of *ccy2*, known as the domestic or quote currency, required to buy one unit of *ccy1*, known as the foreign or base currency, quoted in terms of the domestic currency. The spot rate is therefore equal to the units of *ccy2* per *ccy1*. For example the British pound, US dollar (GBP/USD) quote is for the number of US dollars required to buy one British pound with the price quoted in US dollars. The spot rate, S_t, links the notional amounts that one currency can be exchanged for another. It follows that a factor of S_t notional units of N_f foreign currency is equal to N_d notional units of domestic currency

$$N_d = N_f \cdot S_t \,, \tag{2.1}$$

conversely N_d notional units of domestic currency per the spot rate S_t is equal to N_f units of foreign currency

$$N_f = \frac{N_d}{S_t} \,. \tag{2.2}$$

As spot rates are quoted to finite precision, the last significant digit is called the pip and represents the smallest price increment. The headline figure in the exchange rate, which is often assumed when quoting a rate, is 100 pips. For example if the spot rate for USD/JPY is 113.45, the headline figure is 113 and there are an additional 45 pips. Conversely, if the spot rate for EUR/USD is 1.1336, the headline 100 pips figure is 1.13 and there are an additional 36 pips.

FX instantaneous trades are executed in the FX spot market at the *today date* with the exchange of funds, settlement, happening at the *spot date*, which is most commonly two business days later. The settlement risk associated with these transactions is that one side of these payments does not go through while the other side does. Alternatively an investor long in one currency could exchange a nominal amount at a future date by executing the trade in the FX options market to protect against the depreciation of one currency against another over a period of time. If an FX option with a stated expiry date is exercised at the prearranged strike price then the expiry date is equivalent to the today date and the delivery date equivalent to the FX spot date. Consider an investor holding sterling and intending to buy dollars in six months time. They could buy a put (sell) option on the GBP/USD exchange rate to sell GBP and buy USD in six months time, at a prearranged strike price. This removes the downside risk at the cost of the option premium.

The foreign exchange market is highly liquid. The market is open 24 hours a day from 5:00 *p.m.* EST (9:00 *p.m.* GMT) on Sunday until 4:00 *p.m.* EST (8:00 *p.m.* GMT) on Friday. Trading in FX is not done at one central location, but is conducted between participants by phone and Electronic Communication Networks (ECN) in markets around the world. The FX market has a decentralized multidealer structure. Two implications of decentralisation are fragmentation and lack of transparency: Sager and Taylor (2006).

A highly active interdealer market has developed with institutions located throughout the globe. The share of trading between reporting dealers executing trades and managing risk on their accounts and on behalf of their customers, defined as financial institutions identified by national central banks that report in the Triennial Survey, accounting for 38% of turnover in April 2019, compared with 42% in April 2016. Banks other than reporting dealers accounted for a further 24% of turnover in 2019. Institutional investors were the third largest group of counterparties in the 2019 FX markets, at 12%. Sager and Taylor (2006) identify four types of dealers:

i. Market-makers: facilitating access to interdealer liquidity and providing best execution for customer trades.

ii. Spot or Leverage traders: trade on the basis of the net buy-sell orders executed by the bank's trading desk with an investment horizon of a few hours, or at most days.

iii. Proprietary traders: intra-day traders whose investment time horizon extends from minutes to hours at most.

iv. Senior risk takers: perform a similar function to spot traders at large investment banks, but are allocated a much larger risk budget.

Participants offer to make two-sided quotes to other active dealers in return for receiving the same service. FX dealers in major currencies have access to well-developed interdealer markets and operate in a largely unregulated environment: there is no single regulatory body in this market. Dealer trades are not made public, and no transaction price or volume data for the broad market is available.

Customers interact with dealers in the management of currency exposure. Customer buy-sell orders are the most important source of private information in the foreign exchange market. An understanding of the behaviour of an exchange rate at a given time requires knowledge of the types of customers prevalent in the market at that time and of the ways in which they trade and interact with the wider market. Sager and Taylor (2006) categorised customers as:

i. Passive: inherit foreign exchange exposure from the sale and purchase of underlying assets which are either hedged or unhedged.

ii. Active: add value to underlying portfolios by implementation of appropriate active hedging strategies.

iii. Informed: these include large central banks who observe data relevant to their own currency in advance of other market participants and informed banks who take a bilateral inventory from their customs.

iv. Uninformed: those customers not categorised as informed.

The 2019 Bank for International Settlements (BIS) Triennial Central Bank Survey states that 93 percent of all FX trades taking place are between banks and other financial institutions. Therefore it is reasonable to assume that the net transaction orders a dealer sees from active and informed customers is a small part of the total trade being executed. Orders between banks and other financial institutions are likely to be fed into the market throughout the current trading session. This information will enable a dealer to either consolidate or net-off their trading positions. Participants can execute a trade in one of two ways:

i. Submitting a market order for immediate execution at the best-available current price.

ii. Submitting a limit order for execution at a specified price or better.

Lyons (2001) noted that in the electronically-traded interdealer FX market approximately two thirds of the total spot volume is between dealers dealing on their own accounts: dealer to dealer rather than customer to dealer, thus setting the mid-price that banks use to quote bid-ask spread spot prices to their customers.

2.1.2 Economic Fundamentals and Technical Analysis

Macroeconomic Fundamentals

The Dornbusch (1976) overshooting theory implies that foreign exchange rates will temporarily overreact to changes in monetary policy. This overreaction creates a new equilibrium in the short-term to compensate for the sticky prices of the goods in the economy, which are resistant to change. This means in the short-term the new equilibrium level will be reached through shifts in financial market prices only, not through shifts in the prices of goods themselves. As this resistance to a change in the price of goods gradually diminishes in the long-term, the prices of goods begin to respond to the changes in the financial markets, creating a new long-term equilibrium.

Frankel and Froot (1990) used the overshooting theory of exchange rates to explain some aspects of the movement of the dollar in recent time. They attributed this theory to the dollars movement over the period 1981-1984 when real interest rates in America rose above those of their trading partners, causing the dollar to appreciate significantly. The overshooting theory proposes that as of 1984 the value of the dollar was so far above its long-run equilibrium that expectations of future depreciation were sufficient to offset the higher nominal interest rate in the minds of international investors. Consequently the value of the currency had moved above its long run equilibrium, departing from the value expected from macroeconomic fundamentals. Given that standard macroeconomic variables are not adept at explaining, never mind predicting *ex ante* short term changes in the exchange rate, a possible justification is that the short-term adjustments are rational corrections of the perceived disequilibrium not detected by macroeconomic fundamentals.

An alternative view is the movement is independent of the fundamentals, explained by the existence of a spike in the asset value. This self-confirming market modification, known as a speculative bubble, assumes that all market associates are rational and operate as a single entity. The speculative bubble demands that the market's expectation of the future exchange rate is collectively known by all participants. Participants have homogeneous expectations. However the market shows evidence of heterogeneous expectations. With trading in FX markets averaging $6.6 trillion per day in April 2019, *see* the Bank for International Settlements (BIS) Triennial Central Bank Survey, and with 93 percent of all trades taking place between banks and other financial institutions, the trading volume supports the heterogeneous argument: it is the differences in professionals' expectations, not their commonality that explain why they trade. Taylor et al. (1995) note that the macroeconomic fundamentals are clearly important in setting the parameters within which the exchange rate moves in the short term, but they do not appear to tell the whole story. They note that attempts to provide explanations of short-term exchange rate movements based solely on macroeconomic fundamentals may not prove successful. It is in this context that the emerging literature on foreign exchange market microstructure seems especially promising.

Chartists and Fundamental Analysts

If traders trade by extrapolating recent trends: momentum trading or bandwagon expectations, this can potentially magnify significant swings in the exchange rate. Technical analysts known as chartists, use extrapolative methods such as the spot rate crossing the moving average rate from below as an indicator to buy or from above as an indicator to sell. Conversely analysts that rely on macroeconomic fundamentals are identified as fundamentalists.

Chartists are traders who base their strategies on the analysis and extrapolation of past price movements. Chartists only study the price action of the market; whereas fundamental analysts attempt to look to the reasons behind that action. The chartist supposes that the price contains all the information pertaining to the market: economic and non-economic, rational and irrational, the tensions of demand and supply; therefore the price incorporates the fundamentalist views by default.

Basic chart analysis involves visually identifying recurring patterns in time series price data. Certain configurations, known as *reversal patterns*, are taken to indicate the imminent reversal of a trend. The most famous of these is the 'head and shoulders' formation: Edwards and Magee (2018). Other configurations may be judged to be *continuation patterns*, patterns that occur within established trends: Murphy (1986). Chartists define upper and lower limits of expected price movements, termed *resistance* and *support* levels respectively. Chartists look for automated trend following and non-trend following indicators. A trend following indicator to buy is signalled when a short moving average cuts a longer moving average from below. Conversely a non-trend following indicator might be to price the asset anticipating a market correction, assuming an asset has been overbought or oversold. Clearly, chart analysis has a large subjective element.

Allen and Taylor (1990) looked at the empirical evidence of the impact that chartists or technical analysts have on the London foreign exchange market by conducting a questionnaire survey of the chief foreign exchange dealers. Defining noise traders as dealers who do not base their trading strategies on macroeconomic fundamentals clearly classifies chartists as noise traders. Allen and Taylor (1990) found that the vast majority of traders integrated chartism into their short term (intraday-to-one week) trading strategy. As the forecast timeline extended from one to 12 months the reliance on fundamental analysis became more significant; with a clear shift in favour of fundamental analysis for horizons over 12 months. Speculators use these extrapolative techniques to forecast short term horizons, contrasting theories such as Purchasing Power Parity (PPP) used to forecast a long-run equilibrium.

The question is which is the more prevalent forecasting method; extrapolating short term technical analysis or long term macroeconomic fundamental expectations, and what are the consequences on the market?

The premise behind speculative bubbles suggests that the market has moved away from fundamental analysis and towards technical analysts or chartists in response to the inferior forecasting record of the former. Frankel and Froot (1990) demonstrated support for this theory from the Euromoney yearly survey from 1978 to 1985 of between 10 and 27 foreign exchange trading firms. The survey revealed that the overwhelming majority in 1978 relied exclusively on macroeconomic fundamentals and only two on technical analysis. By 1985 the position had reversed. Only one firm exclusively used fundamentals and the majority using technical analysis. The suggestion is that over time the migration to different forecasting techniques have contributed to large exchange rate movements, which have taken place with little reference to macroeconomic fundamentals. This accounts for the shift toward more purely financial models of exchange rate movements and heightened interest in market microstructure.

Does the move towards microstructure finance explain the FX market, defining a potential pricing model?

2.2 Literature Review

2.2.1 Exchange Rate Dynamics

There have been three approaches to exchange rate determination since the Bretton Woods conference of 1944: *goods* market, *asset* market and *microstructure* market approach.

The pre-1970s goods market approach based its demand for currencies from the purchase and sales of goods. The pre-1970s exchange rates were fixed but adjustable. The rates were negotiated at the Bretton Woods international conference, fundamental to the system overseen by the International Monetary Fund (IMF), and in effect from 1945 until 1971; superseded by floating rates in 1973. Despite its intuitive appeal the data showed that exchange rate movements in foreign exchange (FX) markets are virtually uncorrelated with trade balances. Lyons and Moore (2009) noted that real trade flows or balance of payments, are but a small component of currency transaction flows. It would seem rational to analyse the long-run exchange rate behaviour by considering standard macroeconomic fundamentals, emphasising international trade flows and the accompanied change in the exchange rate for its determination. However, Dornbusch (1976) and Choi (2011) noted that this approach does not distinguish the adjustment speed between the goods and money markets: one lags the other. In essence it is not the gross amount of trading that matters but rather its composition.

The goods market approach was superseded by the asset market approach in the 1970s. Building on the former and recognising that currency demand also came from the purchase and sale of assets. The asset market approach introduced the concept of market efficiency: exchange rates incorporated all publicly available information. Empirical work did not support the asset market approach. R. Meese and Rogoff (1983) showed that the asset market models were no better than 'no change' random walk forecasting models and could not consistently get the direction of change right.

The third approach to exchange-rate determination is the microstructure approach. Although based on the asset market approach: demand for currencies comes from the purchase and sale of assets, what differentiates the microstructure approach is that it relaxes three key assumptions of the asset market, namely:

i. Information: some relevant information is not publicly available.

ii. Players: market participants differ in ways that affect prices.

iii. Institutions: trading mechanisms differ in ways that affect prices.

R. Meese and Rogoff (1983) compared the performance of a microstructure model against a standard macroeconomic model and a random walk. They found that the microstructure model consistently outperformed the alternatives. They concluded that macroeconomic models of exchange rates perform badly at frequencies higher than one year and that fundamental variables are poorly correlated with high frequency exchange rate movements.

Although empirical evidence suggests that standard macroeconomic fundamentals may influence the long-run behaviour of exchange rates, Sager and Taylor (2006) noted that when applied to shorter-term exchange rate models the forecasts quality proved to be perilous. Indeed R. Meese and Rogoff (1983) and R. A. Meese (1990) state that the explanatory power of such macroeconomic models is essentially zero.

Given that traditional macroeconomic approaches are not consistent with high frequency data, Flood and Taylor (1996) noted that our understanding of the short run behaviour of exchange rates is unlikely to be enhanced by further examination of the macroeconomic fundamentals. Flood and Taylor (1996) advocated that a new work on the microstructure of the foreign exchange markets seemed both warranted and promising. This thesis will concentrate on the microstructure determinants of the FX spot rates.

Evans and Lyons (2002) proposed a microstructure approach, highlighting variables that play no part in macroeconomic modelling but become integral to microstructure finance. The most important of these variables are *order flow* and *bid-ask spread*. Therefore in questioning the microstructure approach to understanding exchange rate behaviour this thesis asks the following question: does the microstructure approach offer an explanation of the FX market and avail itself to a model?

Microstructure Variables: Order Flow

The quality of macroeconomic fundamental based exchange rate forecasts can be improved by the microstructure measure of order flow. Sager and Taylor (2006) inferred that order flow allows the wider market to learn about the private information and trading strategies of better informed participants. Order flow is defined as the net of positive: buyer-initiated and negative: seller-initiated orders and seen as a measure of net (positive or negative) buying pressure. A positive (negative) sum means a net buying (selling) pressure. Order flow is closely related to price in these models because of the information it conveys to the market and can be regarded as the transmission mechanism that maps information to price. Evans and Lyons (2002) explain that order flow employs public information regarding:

i. The expectations of the participants.

ii. Transaction standings.

iii. Macroeconomic fundamentals.

Accepting that order flow and nominal exchange rates are strongly correlated, Evans and Lyons (2002) determined that interdealer order flow drives the exchange rate via information aggregation. They noted that prices increased with an increase in net buying pressure. The intuition is that uncertain public demand for foreign exchange is realised at the start of each day. This produces orders that are not publicly observable so the information these orders convey is aggregated into the trading process. These demands affect price because the rest of the market, being less than perfectly elastic, require a price concession to absorb them.

Lyons (2001) considered how microstructure theory emphasises the role of order flow in determining the price of foreign exchange (FX) spot trades within a large bank. Lyons (2001) noted that despite his presumption that exchange rates depended on macroeconomic variables: inflation, output, interest rates, *etc.* only rarely was news of this type a primary concern to FX traders. This highlighted a gap between macroeconomics exchange-rates on the one hand and microstructure finance on the other. Frankel and Rose (1995) go further and state that there is little evidence that macroeconomic variables have consistent strong affects on floating exchange rates. One can draw a logical conclusion that macroeconomic fundamentals do not lend themselves to modelling the high frequency FX market.

Microstructure Variables: Bid-Ask Spread

The spread is a transaction cost. Liquidity theory affirms that the bid-ask spread is increased by a rise of transaction cost and volatility and inversely associated with liquidity, order flow and major currency pairs: Sarkissian (2016). The bid-ask spread is the difference between the highest price that a buyer is willing to pay for an asset and the lowest price that a seller is willing to accept to sell it. The FX market requires no legal maximum of the bid-ask spread range.

The field of microstructure finance sought to separate itself from rational expectation models which abstract from the trading mechanisms. The premise being that trading mechanisms have little affect on the relationship between the fundamentals and price. Microstructure finance raises the question of how does modifying an aspect of the trading mechanism: dissipating information and transaction costs; alter the price in an order driven market? This question addresses the attributes of the bid-ask spread and their impact in determining the market FX rates.

Bid-Ask Spread Determinants

Huang and Masulis (1999) considered the effects of dealer competition in determining the bid-ask spread in the spot foreign exchange market. Dealer activity in the market is appraised by the number of dealers entering quotes per 15-minute period. Dealer activity is associated with market depth and the ease with which dealers can offload unwanted inventory. Huang and Masulis (1999) found that bid-ask spreads decrease with an increase in dealer competition. Conversely, adversary theory suggests that an increase in the exchange rate volatility causes spread risk to increase. The reduction in order flows and volume will also widen the spread.

Huang and Masulis (1999) interpreted the association of time-varying bid-ask spreads and the number of active dealers as a short-run equilibrium outcome resulting from random deviations of the long-run level of net customer transactions and dealer entries and exits. Such deviations are driven by profit maximization. Huang and Masulis (1999) suggest that in the short run, a variation in the number of dealers is caused by:

i. The behavioural characteristic of customer order flow.

ii. Current dealer inventory positions.

iii. Current and near term market depth.

iv. Inventory risk considerations.

Previous studies of FX spreads focused on the effect of exchange rate volatility: presumed to impact bid-ask spreads by positively affecting dealer risk bearing. The differing market microstructure models of spread determination show that competition is a prime component of bid-ask spreads. Ho and Stoll (1983), Biais (1993) and Glosten (1994) found that as the number of dealers rises, the bid-ask spreads of individual dealers fall because the expected best bid and ask quotes of rival dealers are more competitive. Dealer spreads also can vary if dealers have differential access to information on the state of the market: large banks tend to quote more aggressively than small banks due to their information advantages. Huang and Masulis (1999) found that FX bid-ask spreads culminate in the interaction between two factors:

i. Institutional: due to the structure of the FX market and informational advantage.

ii. Behavioural: due to dealer competition, dealer quotes and inventory adjustments.

If a dealer does not want to increase their position materially, then less aggressive bid and ask quotes are offered, which result in a higher spread. Huang and Masulis (1999) concluded that dealer competition is an important determinant of bid-ask spread.

In breaking down the constituents of the bid-ask spread; whether a given trade is transacted near the prevailing bid quote or the prevailing ask quote, McGroarty, ap Gwilym, and Thomas (2007) looked at the trade indicator models. These models relate the time series of returns to the side of the trades. Trades oscillating between the two sides identify the spread and enable its measurement. The most general spread decomposition model is given by Huang and Stoll (1997). This model was built around the equity based, market-maker centric NYSE that does not conform to the electronic interdealer order-driven spot FX market. McGroarty et al. (2007) revised the specification of the bid-ask spread decomposition model to take account of particular features of electronic order driven markets. Limit order driven bid-ask spreads are determined by the risk of the orders not being executed. When the risk is low the spread is wide, traders then submit limit orders in preference to market orders, narrowing the spread. In an order driven market non-execution risk is lower when the volume is high and volatility is low. However order-driven microstructure literature assumes that informed traders chose to submit market orders in preference to limit orders. Bloomfield, O'Hara, and Saar (2005) found the reverse to be true, traders are more likely to submit limit orders than market orders. Huang and Stoll (1997) estimated a basic trade indicator model of spread components, noting that three variables make up the general decomposition models:

i. The unobservable underlying value of stock in the absence of transactions costs, V_t, determined just prior to the posting of bid and ask quotes at time t.

ii. Midpoint quote, MP_t, from the prevailing bid and ask quotes before a transaction.

iii. The buy-sell indicator variable: order flow, of_t, for the transaction price, TP_t.

McGroarty et al. (2007) revised the model for the electronic order driven markets. McGroarty et al. (2007) argued if informed traders are setting prices, the underlying value V_t is solely determined by public information shocks: ε_t, given by

$$V_t = V_{t-1} + \varepsilon_t \ \text{ where } \ \varepsilon_t \sim_{\text{iid}} N(0, \sigma^2)\,. \tag{2.3}$$

An interim variable V_t^*, representing the disturbed value of V_t by a buy-sell imbalance, consistent with the mechanism of an order-driven market is

$$V_t^* = V_t + \beta\,\frac{S_d}{2}\sum_{t=1}^{t-1} of_t\,, \tag{2.4}$$

where of_t is the order flow trade indicator variable and takes the value $+1$ when the transaction is initiated by the buyer, -1 when initiated by the seller and 0 when neither party can be identified as initiator. The spread is S_d and thus $\frac{S_d}{2}$ is the half spread and β is the temporary buy-sell imbalance component. The midpoint quote, MP_t, can now be identified as a function of V_t^*

$$MP_t = V_t^* - \alpha^*\,\frac{S_d}{2}\,of_t\,, \tag{2.5}$$

where α^* is the private information component. The choice facing every trader is whether to submit a limit order or a market order. Aggressive buying and selling by an informed trader translated into aggressive limit orders will narrow existing bid-ask spreads, enticing traders on the opposite side to submit market orders over limit orders. Thus an upward revision of the price triggers a sell, producing a negative relationship between MP_t and of_t. Thus the McGroarty et al. (2007) modified trade indicator model becomes

$$\Delta TP_t = (1-\alpha^*)\,\frac{S_{dt}}{2} of_t + (\alpha^* + \beta - 1)\,\frac{S_{dt-1}}{2}\, of_{t-1} + e_t\,, \tag{2.6}$$

where ΔTP_t is the return defined as the change in transaction price and of_t is order flow. The error term, e_t, combines public information releases which effect prices with random deviations in the bid-ask spread. The trade indicator model relates return to order flow and the residual component of the bid-ask spread. The residual accounts for the private information and temporary buy-sell imbalance factors, $(1-\alpha^*-\beta)$, and is attributed to price clustering.

Trading volume and order flow have both been closely associated with informed trader activity in the market microstructure literature. Using theory that explains regular intraday trading patterns to transform trading volume and order flow into proxies for private information McGroarty, ap Gwilym, and Thomas (2009) examined the relationship between bid-ask spreads and return volatility. Private information is a common theme of intraday trading data patterns. These trading patterns are identified in volume, volatility and bid-ask spreads. To explain such phenomenon McGroarty et al. (2009) explored how private information influences price changes and bid-ask spreads. McGroarty et al. (2009) introduced order flow as an additional variable determining intraday behaviour analysis.

Intraday trading data exhibit M and U shaped patterns in bid-ask spread, trading volume, return volatility and order flow during the opening, trading normally and closing day processes. McGroarty et al. (2009) reviewed the role of private information to explain bid-ask spreads and exchange rate volatility. McGroarty et al. (2009) noted that the volatility contributes to incremental returns which determine the overall price level.

The vast majority of intraday patterns of bid-ask spread are U shaped which show wide spreads at the market opening and closing and moderate levels in between. Conversely trading volume and return volatility are M shaped patterns with small volumes and low volatility at the market opening and closing and moderate levels in between. However the literature on intraday empirical patterns neglects intraday order flow. Evans and Lyons (2002) argue that order flow is driven by private information and a driver of price in the spot FX market. An explanation of these intraday patterns is attributed to either:

i. Differing trader behaviour at the opening and closing.

ii. The strategic behaviour of informed traders.

Admati and Pfleiderer (1988) noted that informed traders trade when uninformed traders are present in the market to minimise their transactions costs: bid-ask spreads. This explanation is not sufficient to treat volume as a proxy for private information. Any relationship between high volume and bid-ask spreads or return volatility may just as easily be caused by high, uninformed trading volume as by informed trading volume. Easley and O'Hara (1992) suggested a link between trading volume and private information. Private informal signals cause trading volume to deviate from its normal levels. This deviation splits the trading volume into unexpected components attributed to private information and expected components which are not.

2.2.2 Theoretical Models of the Foreign Exchange Rate

Having looked at the structure of the foreign exchange market and described the various participant groups, now consider how closely the traditional asset-price and microstructural exchange rate models reflect this configuration.

Traditional Asset-Price Models

R. A. Meese and Rose (1991) examined the empirical relation between nominal exchange rates and macroeconomic fundamentals in assessing the non-linearities: where changes in the output do not change in direct proportion to changes in any of the inputs. Schinasi and Swamy (1989) suggested that taking account of non-linearities may lead to improved forecasting ability of exchange rate models.

In assessing the non-linearities, five theoretical models of exchange rate determination are considered. The first three are variants of the monetary models of Dornbusch (1976), Frenkel (1976) and Mussa (1977). The models consist of domestic and foreign currency demand equations with a stationary disturbance: an equation relating the expected change in the spot rate to the interest rate differential and an exogenously-varying risk premium on domestic assets. The first of the five models: the flexible-price monetary model, assumes Purchasing Power Parity (PPP) holds up to an exogenous real exchange rate shock and represented by

$$S = f(m, ip, ir) + \varepsilon_t \text{ where } \varepsilon_t \sim_{\text{iid}} N(0, \sigma^2)\,, \tag{2.7}$$

where S is the spot exchange rate; m is the ratio of domestic to foreign nominal money supply and ip is relative industrial production and ir is the nominal interest differential. The next two are sticky-price monetary models: prices cannot be changed easily even if there are changes in other sectors of the economy. One version does not contain cumulated domestic and foreign trade balances, while the other does. Both sticky-price variants assume slow adjustment of goods prices relative to asset prices, and thus allow deviations from PPP. The sticky-price models are represented by

$$S = f(m, ip, ir, p, tb) + \varepsilon_t \text{ where } \varepsilon_t \sim_{\text{iid}} N(0, \sigma^2)\,, \tag{2.8}$$

where p is the inflation differential and tb the relative cumulated trade balances. Note that the inflation differential, p and trade balances, tb, do not enter Equation (2.7) the flexible-price monetary model.

The first sticky-price monetary model imposes a constraint on trade balances, tb, and assumes that the real interest differential: the nominal interest rate minus the expected rate of inflation, $ir - p$, is an appropriate explanatory variable.

The second sticky-price monetary model also employs the real interest differential, but has no restriction on the trade balance term.

The second group of exchange rate models considered is based on explicit maximising behaviour.

The first is an adaptation of the highly-stylised Lucas (1982) model of a two-good, two-country, pure-exchange economy. A representative agent who consumes both foreign and domestic output maximises the expected discounted utility of current and future consumption subject to budget and cash-in-advance constraints. The solution for the spot exchange rate is the product of relative monies, incomes and the marginal rate of substitution between domestic and foreign goods. Taking a Cobb-Douglas utility: a workable form of the production function used to represent the technological relationship between the amounts of two or more inputs and the amount of output that can be produced by those inputs, the model is parameterised. This in turn implies that the spot exchange rate can be simply related to relative money supplies and domestic outputs. This model is represented by

$$S = f(m, ip) + \varepsilon_t \text{ where } \varepsilon_t \sim_{\text{iid}} N(0, \sigma^2)\,. \tag{2.9}$$

The fifth model is the Hodrick (1988) extension of the Svensson (1985) price stability exchange rate. The basic framework is that of Lucas (1982) with a modification of the timing of goods and money market transactions. The Hodrick (1988) contribution is to add exogenous fiscal policy and examine the effect of time-varying conditional variances of the exogenous processes on the level of the spot rate. This model is represented by

$$S = f(m, ip, \delta m, h(m), h(ip)), h(\delta m) + \varepsilon_t \text{ where } \varepsilon_t \sim_{\text{iid}} N(0, \sigma^2)\,, \tag{2.10}$$

where δm is the change in relative money growth rates and $h(-)$ is the conditional variance of the variable in parentheses.

In assessing the importance of non-linearities in empirical models of exchange rates R. A. Meese and Rose (1991) resolved that the poor explanatory power could not be attributed to non-linearities arising from an improper functional form. That is the performance could not be attributed to the misspecification of utility, production or the demand for money functions in standard linear models. Thus incorporating non-linearities into existing structural model does not improve our ability to understand how exchange rates are determined.

Sager and Taylor (2006) present a general representation of the asset-price models of exchange rate determination and takes the form

$$S_t = \beta' F_t + S^e_{t+1}\,, \tag{2.11}$$

where S_t is the spot exchange rate at time t, S^e_{t+1} is the one period ahead expected spot rate given information available at time t, F_t is the vector of macroeconomic fundamental variables that exhibit explanatory powers for the determination of exchange rates and β' is the vector of factor loadings: the degree to which a factor acts upon a variable in the process. The range of loadings is between -1 to 1. Loadings which are closer to 1 or -1 show that the factor has a strong effect on the variable whereas, the loadings which are closer to zero show that the factor weakly effects the variable. This approach is sufficiently general to encompass simple monetary models, sticky-price overshooting models and portfolio balance models.

Under the assumption of rational expectations, decisions based on human rationality, past experience and new information made available by the changes in the macroeconomic fundamentals are instantly incorporated into the exchange rates as soon as they are released. Thus S^e_{t+1} represents the true conditional mathematical expectation of the one period ahead spot rate. This implies homogeneous expectations of the market participants whereas it is their heterogeneity that causes participants to trade. This model is independent of the factors identified by Huang and Masulis (1999) that determine the bid-ask spread, namely the way information is processed or the institutional structure of the market.

Asset-price models are equilibrium models distinct from the method used to reach that equilibrium state. The models tell us nothing about how information on macroeconomic fundamentals gets compounded into the current exchange rate and ignores institutional structure. What is accepted is that macroeconomic fundamental variables: inflation, output, interest rates, *etc.* are measured at a low frequency; monthly or quarterly and are often subject to error and revisions. This is possibly the reason why asset-price models have performed poorly in explaining short-run exchange rate movements.

The Evans-Lyons Microstructural Model

Lyons (2001) and Evans and Lyons (2002) noted a microstructural approach to exchange rate determination is not necessarily at odds with asset-price models. While traditional asset-price exchange rate models may tell us something about equilibrium conditions or fair value, they ignore aspects evident in microstructure. Traditional models tell us nothing about the information transmission mechanism and institutional structure.

In assessing the relationship between the information transmission mechanism and the institutional structure consider the market structure as comprising three stages during the trading day:

i. Stage One: Having observed the return, R_t, at the start of each day, dealers independently and simultaneously set bid-ask spreads for, and trade with, customers. Customers throughout this process are non-reporting dealers and include banks: other than reporting dealers, and institutional investors.

ii. Stage Two: Dealers trade amongst themselves, independently and simultaneously posting a bid-ask spread for other traders. These spreads lead to trades, as dealers spread risk generated by earlier customer trades through the interbank market. Once trading is complete dealers are able to observe the order flow that has occurred. This is assumed to convey information about customers trading in stage one.

iii. Stage Three: Dealers again trade with customers to share overnight risk more widely across the market; supposing customers willingly absorb dealer's inventory imbalances. Dealers do not run overnight open positions.

Hence the closing exchange rate, CP_t, at the end of each day within the Evans and Lyons (2002) model will be

$$CP_t = \sum_{t=1}^{T} R_t + \sum_{t=1}^{T} of_t \,, \tag{2.12}$$

where R_t is the dealers periodic return attributed to the flow of macroeconomic information, and of_t is order flow. The change in the closing exchange rates, ΔCP_t, from the end of day $t-1$ to the end of day t can be written as

$$\Delta CP_t = \Delta R_t + \Delta Tof_t \,, \tag{2.13}$$

where Tof_t is the total order flow generated by interdealer trading during the day. A distinction in the model exists between *active* and *passive* customers. Stage one customers are active: implementing appropriate strategies they generate initial imbalances in dealers' positions. While stage three customers are passive: inheriting foreign exchange exposure by passively absorbing dealers' positions at the end of the day.

Although the Evans and Lyons (2002) suggested dealer-customer relationship is useful, the notion that customers willingly absorb the daily inventory imbalance of dealers seems at odds with market practice. As the width of bid-ask spreads is a positive function of transaction size, dealers divide customer trades into smaller tranches as they spread risk through the interbank market, thus saving transactions costs.

2.2.3 Arbitrage Equilibrium for the Triangulation Transmission Mechanism

The market microstructure allows us to explain the short-run exchange rate movement in terms of order flows and bid-ask spreads. Evans (2002) noted that order flows employ public information regarding the expectations of the participants, the transaction standings and the macroeconomic fundamentals, whereas the bid-ask spread communicates the volume and volatility of the market. In reviewing the microstructure behaviour of the FX market it is noted that:

i. The information effect of order flow increase prices.

ii. Liquidity theory inferring that the bid-ask spread is increased by a rise of transaction costs or a reduction in order flows.

iii. Adversary theory suggesting that volatility causes spread risk to increase.

Now consider the significance of arbitrage equilibrium in the FX market. What is the effect of the interaction between FX exchange rates caused by currency cross rates and currency cycle triangular arbitrage transactions in the FX market? How, if at all, is this linked to order flow and the bid-ask spread?

Choi (2011) attempted to explain exchange rate dynamics by focusing on the arbitrage and the spread in triangular foreign exchange trades. Choi (2011) noted that there is only one possible triangle when a bilateral exchange rate equals its trilateral rate, otherwise any third currency would potentially create arbitrage opportunities. To explain exchange rate movements Choi (2011) assumed the existence of three levels of time dependent determinants:

i. Exchange rates are identified by the macroeconomic fundamental drivers at the long-term equilibrium.

ii. Short term exchange rates exhibit prices which deviate from macroeconomic fundamentals.

iii. The existence of very short-run, momentary, exchange rates before the markets function efficiently to cancel them out.

The market drives the bilateral and trilateral exchange rates to equal each other by arbitrage. Moosa (2001) argues that triangular arbitrage maintains the market equilibrium condition at which point the rates are consistent and there is no profitable arbitrage opportunities. Moosa (2001) found the effect of triangular arbitrage in the forward market is equivalent to the combined effect of covered interest arbitrage and triangular arbitrage in the spot market.

The equilibrium state is accepting that the cross rate of a currency with other currencies, and those currencies with each other are equalised through triangular arbitrage. The triangular arbitrage is a financial activity that takes advantage of three exchange rates. Osu (2010) noted that arbitrage is not a perfect equaliser because the market is not perfectly efficient. Aiba, Hatano, Takayasu, Marumo, and Shimizu (2002) noted that arbitrage opportunities existed and generate an interaction among foreign exchange rates. These interactions result in the auto-correlation function of foreign exchange rates being negative in a short time scale. When examining the triangular activity of three exchange rates, Aiba et al. (2002) noted that the product of the three foreign exchange rates converges to its average. Given exchange rates AB, AC and CB, *see* Table 3.1 on page 28, a trader exchanges one unit of currency A for an amount of currency B and that amount of B for an amount of currency C and finally back to currency A, the final amount of A realised is given by

$$\upsilon \equiv \prod_{i=1}^{3} S_i(t) = AB \cdot \frac{1}{AB} \cdot \frac{1}{AC} \approx 1\,, \tag{2.14}$$

where υ is the rate product and $S_i(t)$ is the exchange rate for currency i at time t. To be precise, there are two types of the rate product. One in the direction outlined above. The other is based on the transaction in the opposite direction. The two values show similar behaviour. This paper will focus on the first type. If υ, the rate product, is greater than unity a profitable triangular arbitrage opportunity exists. Hence many traders will immediately complete this transaction, moving the rate product υ below unity and eliminating the arbitrage. Arbitrage opportunities arise because the exchange rate $S_i(t)$ fluctuates strongly. Aiba, Hatano, Takayasu, Marumo, and Shimizu (2003) define the logarithm rate product ln υ as ν' given by

$$\nu'(t) = \ln \prod_{i=1}^{3} S_i(t) = \sum_{i=1}^{3} \ln S_i(t)\,. \tag{2.15}$$

There is a triangular arbitrage opportunity whenever this value is positive.

To study the effects of triangular arbitrage on the fluctuations of the exchange rates, Aiba et al. (2003) constructed a stochastic model of the time evolution of exchange rates. The model's interaction accounts for the effect of the triangular arbitrage transactions. The basic equation is the time evolution of the logarithm of each rate, given by

$$\ln S_i(t+T) = \ln S_i(t) + f_i(t) + g(\nu'(t)) \qquad (i = 1, 2, 3)\,, \tag{2.16}$$

where T is a time step which controls the time scale of the model, f_i denotes independent fluctuations that obeys a truncated Levy distribution: a continuous probability distribution for a non-negative random variable and g represents an interaction function denoted by

$$g(\nu'(t)) = -a(\nu' - \bar{\nu}')\,, \tag{2.17}$$

where a is a positive constant which specifies the interaction strength and ν' is the time-evolution of the logarithm rate product, where $\bar{\nu}'$ is the time average of ν'. The time-evolution of the logarithm rate product is given by

$$\nu'(t+T) - \bar{\nu}' = (1 - 3a)(\nu'(t) - \bar{\nu}') + F(t)\,, \tag{2.18}$$

where

$$F(t) \equiv \sum_{i=1}^{3} f_i(t)\,. \tag{2.19}$$

The model successfully described the fluctuation of the data of the real market. Aiba et al. (2003) noted that the rate product, v, fluctuates close to unity. Plotting the probability density function of the rate product, Aiba et al. (2003) illustrated that the function had a sharp peak and fat-tails while the probability density function of the three constituent exchange rates, $S_i(t)$, did not. Aiba et al. (2003) noted that this model describes a fat-tailed probability distribution of the actual market.

The triangular arbitrage transaction is a financial activity that takes advantage of the fluctuations in three foreign exchange rates among three currencies: the product of the three foreign exchange rates converge to the rate product, thereby generating an interaction among the rates. Aiba and Hatano (2006) highlight another consequence of the triangular arbitrage interaction: a correlation among exchange rates that makes the rate product converge to its average. The positive constant a is related to the auto-correlation function of ν' as

$$(1 - 3a) = \frac{Av(\nu'(t+T) \cdot \nu'(t)) - Av(\nu'(t))^2}{Av(\nu'(t)^2) - Av(\nu'(t))^2}\,. \tag{2.20}$$

Aiba et al. (2003) noted that as a result of the disequilibrium of triangular arbitrage in a short time scale: before participants enter the market restoring equilibrium, the auto-correlation of each exchange rate in that short time scale is negative. Aiba and Hatano (2006) refers to this as a macroscopic model of triangular arbitrage and regard it as a one-dimensional random walk of three exchange rates, $\ln S_i$, with a restoring equilibrium. Given the logarithm rate product ν' is the sum of $\ln S_i$, Aiba and Hatano (2006) regard ν' to be the average equilibrium value the variables converge to, driven by the interaction function $g(\nu')$.

Describing the microscopic interactions among foreign exchange markets Aiba and Hatano (2006) introduced a new model focusing on the dynamics of each dealer in the market based on the Sato and Takayasu (1998) dealer model. Although the focus is on the interactions among three currencies, two of the three markets can be regarded as one effective market for the purpose of the microscopic model. The basic assumption of the Sato and Takayasu (1998) model is that dealers want to buy currencies at a lower price and to sell currencies at a higher price. There are N dealers; the ith dealer bidding to buy at $B_i(t)$ and sell at $\bar{B}_i(t)$. Making the simplifying assumption that the difference between the buying and selling price is a constant: $\Lambda_p \equiv \bar{B}_i(t) - B_i(t) > 0$ for all i. The model assumes that a trade takes place between the dealer who proposes the maximum buying price and the one who proposes the minimum selling price. Transactions occur when

$$Max\{B_i(t)\} \geq Min\{\bar{B}_i(t)\} \quad \text{or} \quad Max\{B_i(t)\} - Min\{\bar{B}_i(t)\} \geq \Lambda_p \,, \tag{2.21}$$

where the logarithm of market spot rates correspond to $Max\{B_i(t)\}$ and $Min\{\bar{B}_i(t)\}$ denoting the maximum and minimum values in the dealers buying threshold $\{B_i(t)\}$. The foreign exchange spot rate, $S_i(t)$ is defined by the mean of $Max\{B_i\}$ and $Min\{\bar{B}_i\}$ when the trade takes place. Dealers change their prices by the following deterministic rule

$$B_i(t+1) = B_i(t) + a'_i(t) + c'\Delta S_i(t) \,, \tag{2.22}$$

where $a'_i(t)$ denotes the ith dealers price movement at time t: positive when the ith dealer is a buyer and is negative when the dealer is a seller. Once the transaction takes place, the buying dealer changes the sign of $a'_i(t)$ from positive to negative and the selling dealer changes it from negative to positive. $\Delta S_i(t)$ is the change in the exchange rate at time t, $B_i(t)$ is the rate abounding at the previous trade and $c' > 0$ is a constant specifying the dealer's response to changes in the exchange rate, common to all dealers in the market. Thus the Sato and Takayasu (1998) model specifies four parameters:

i. The number of dealers, N.

ii. The spread between the buying price and the selling price, Λ_p.

iii. The dealers response to the market change, c'.

iv. Dealers characteristic movements in a unit time, $a'_i(t)$.

In order to reproduce the effects of triangular arbitrage, Aiba and Hatano (2006) applied the Sato and Takayasu (1998) model twice. They combined two of the three markets as one effective market. Aiba and Hatano (2006) found both the macroscopic and microscopic models reproduced the interaction among the markets. The models successfully demonstrated the actual behaviour of the logarithm rate product ν'. However the microscopic model described more detail than the macroscopic model: the skewness of the distribution of the logarithm rate product ν'. The arbitrage equilibrium state clearly describes the dynamic functionality of the market. Does this functionality lend itself to deriving a model of the FX market?

2.3 Conclusions

In defining the functionality of the high frequency FX market this thesis endorses, a priori, a microstructure rather than a macroeconomic approach. The microstucture variables of order flow, which conveys the net buying pressure, and the bid-ask spread, conveying the transaction costs, form the elements of the information transmission mechanism that determines the FX spot price. This chapter advocates, a priori, that triangulation, which transmits information about the spot price, is also the method of conveying the order flow and bid-ask spread information to market practitioners. Although each of these elements conveys information unique to their own function, this thesis posits that the elements also combine with triangulation to form the systemic attributes of the pricing information flows. The thesis concludes that although each of these attributes fulfils a unique role, it is clear that in order to model the FX market behaviour these attributes need to be considered as operating as a system. The thesis postulates that the FX market behaves as a system of attributes conveying the market information to practitioners. Therefore the information transmission mechanism, consisting of order flow, bid-ask spread and triangulation, is the determining factor in deriving the FX spot price.

As the market drives bilateral and trilateral exchange rates to equal each other by exploiting an arbitrage opportunity, the information flow effecting the arbitrage opportunity is determined by the interdependency between order flow, bid-ask spread and triangulation. Whereas order flow and triangulation are clearly defined, this thesis has focused on the constituent elements that make up the microstructure variable bid-ask spread, which requires greater examination. The explanatory factors that make up the bid-ask spread are introduced within the Lyons (2001) framework and shown to be important characteristics to understand the composition of the spread. Thus the elements that make-up the bid-ask spread, *i.e.* price discreteness and price clustering, are examined in the following chapter. This thesis proposes that in modelling the FX market, these attributes and how they function as a system warrants further scrutiny.

Chapter 3

Triangulation Discreteness Clustering

The microstructure variable described as an information transmission mechanism for order flow, apprises the wider market about the private information and strategies of informed traders: Evans and Lyons (2002) and Sager and Taylor (2006). Rather than simply regarding order flow as the transmission mechanism, this is more nuanced and also incorporates the information being conveyed. Inherent in the net buying pressure is information about trading positions, customer interactions and price.

Dealers seek to use any embedded information which affords a trading advantage that could be used profitably in the markets. Information flow is an essential attribute of the market. Order flow, bid-ask spread and triangulation are the three elements of information flow. There is a suggestion that they are closely linked and combine to describe the FX market behaviour. Individually they contain information regarding their own sphere:

i. Triangulation communicates exchange rate information: the bilateral rate must equal the trilateral rate in equilibrium, Aiba et al. (2002) and Choi (2011).

ii. Order flow communicates the transaction standing and the macroeconomic fundamentals.

iii. The bid-ask spread communicates volume and volatility of the market, Evans (2002).

Collectively triangulation communicates:

i. A price shock to one exchange rate to the two other pairs: McGroarty et al. (2009).

ii. The combined direct and indirect spread differentials: McGroarty, ap Gwilym, and Thomas (2006).

iii. The order flow in one currency pair to explain the rate in another pair: Danielsson, Luo, and Payne (2012).

A suggested area of further research is to look at the three elements of information flow and how they operate as a system.

This chapter considers what information is contained in the bid-ask spread and how this and the order flow information is revealed by triangulation in describing the market. The focus of this chapter is on the attributes of a key variable in the microstructure approach to modelling the FX market: the distinguishing qualities of price discreteness and price clustering. The objective is to quantify the significance and scale of these features and determine how they are interpreted and communicated in the market. The features of the bid-ask spread reveal how the elements of information flow describe the FX market.

3.1 The Composition of the Bid-Ask Spread

The delineation of price discreteness and price clustering are important to understand the behaviour of bid-ask spreads in the spot FX market. McGroarty et al. (2007) looked at price clustering to understand the composition of the bid-ask spreads and how they relate to the spot FX market. In this market the bid-ask spread arises as the difference between the best limit order bid price and the best limit order ask price that are ready to trade. Redefining the technical elements of the decomposition model, *see* Equation (2.6) on page 17, McGroarty et al. (2007) identified three key components of the bid-ask spread:

i. Private information: informed traders move the price and it does not return to its pre-traded level.

ii. Transaction volume: temporary imbalances in the buying and selling volumes are quickly restored.

iii. Residual bid-ask spread: after addressing points *i* and *ii*, this was previously allocated to order processing and administration costs but a more appropriate interpretation of the residual factor is price clustering.

Price clustering refers to the fact that not all the available digits are used equitably. Goodhart and Curcio (1991) noted that rates ending in 0 or 5 tend to be used more frequently. In the FX market exchange rates are truncated to a fixed number of digits, referred to as price discreteness. For clarity the terms prices and exchange rates are inter-changeable throughout this work. Prices are not infinitely long, rather specified to five digit accuracy. How are the microstructure bid-ask spread elements that relate to price: discreteness and clustering, correlated to the exchange rate? To what extent are discreteness and clustering present in the FX market and what is their effect on bid-ask spreads?

The remainder of this chapter is organised as follows. Section 3.2 discusses the information flow due to triangulation and the importance of price discreteness and price clustering to the bid-ask spread. Section 3.3 discusses the data set and empirical methodology. Section 3.4 presents the results and 3.5 the conclusions.

3.2 Triangulation, Price Discreteness and Clustering

Triangulation

In the electronic intraday spot FX market, bid-ask spreads are different from ones suggested by market maker centric models proposed by theory. McGroarty et al. (2006) noted that the absence of market makers will cause a temporary price disturbance due to an imbalance between buyers and sellers. This imbalance contributes to higher volatility at higher-frequencies. Traders submitting aggressive limit orders cause the bid-ask spreads to narrow, inducing transactions to oscillate between the bid and ask quote, contributing to return volatility. Hence there should be a positive, but weak, correlation between bid-ask spreads and return volatility.

There is a feature of the interdealer spot FX market that is important in determining the bid-ask spread and volatility. The cross exchange rate arbitrage. McGroarty et al. (2009) refer to this as a vehicle currency transaction: agents engage in currency trades indirectly using the US dollar instead of direct bilateral trade among their own currencies. Choi (2011) examined the exchange rate dynamics in terms of arbitrage and the bid-ask spread in triangular FX trades. Choi (2011) noted there is only one possible triangle that can exist when bilateral exchange rates equal the trilateral rates; otherwise a third currency would create an arbitrage opportunity. Any bilateral deviations from a common currency side of the trilateral rates will be eliminated in an efficient market, restoring equilibrium. To explain these features consider as an example an exchange rate system with three currencies: A, B and C. These give rise to three exchange rates: AB, AC and CB, *see* Table 3.1.

TABLE 3.1: Example Currency Pairs Bid and Ask Quote

Currency Pair	Bid	Ask
AB	1.1745	1.1745
AC	0.8782	0.8783
CB	1.3373	1.3374

Source: City Index, December 2018.

A buy trade involves buying the foreign base and selling the domestic quote moving from *quote* to *base* currency, therefore dividing the nominal by the *ask* spot rate. Conversely a sell trade requires selling the foreign base and buying the domestic quote, moving from *base* to *quote* currency, hence multiplying the nominal by the *bid* spot rate. A trader with one unit of A to invest can attempt the following arbitrage process $A \to C \to B \to A$:

1. Invest $A1$ into C by selling at the AC bid rate: $A1 \cdot 0.8782 = C0.8782$
2. Invest $C0.8782$ into B by selling at the CB bid rate: $C0.8782 \cdot 1.3373 = B1.1744$
3. Invest $B1.1744$ into A by buying at the AB ask rate: $\dfrac{B1.1744}{1.1745} = A0.9999 \approx A1$

Alternatively the same trader can attempt the arbitrage process $A \to B \to C \to A$:

1. Invest $A1$ into B by selling at the AB bid rate: $A1 \cdot 1.1745 = B1.1745$
2. Invest $B1.1745$ into C by buying at the CB ask rate: $\dfrac{B1.1745}{1.3374} = C0.8782$
3. Invest $C0.8782$ into A by buying at the AC ask rate: $\dfrac{C0.8782}{0.8783} = A0.9999 \approx A1$

At equilibrium no arbitrage opportunities exist: the trader started with $A1$ units and the investment returned $A1$ units. What is the impact of triangulation on bid-ask spreads and order flow?

Suppose an exogenous shock caused the AC currency pair to rise by 10%. If there is no change of the like in the other two rates, a riskless arbitrage opportunity arises. To restore equilibrium, thereby extinguishing the riskless profit opportunity, the exogenous shock affecting one exchange rate must be transferred to other exchange rates.

In addition to the information flow due to price, the triangulated relationship communicates the effect due to an imbalance in the bid-ask spreads.

If the combined indirect spread, for example AB & CB, is lower than the direct AC spread, the volume will move away from the direct currency pair AC to the indirect combination, AB & CB, adversely impacting the direct pair and re-establishing the triangulated process to restore the bid-ask spread equilibrium.

When considering the triangular arbitrage cross-market effect of order flow, Danielsson et al. (2012) noted that the order flow in one currency pair could be used to explain the exchange rate of a second currency pair. When presented with triangular currency pairs an informed trader can use any currency pair to exploit the information advantage. Order flow in one pair might drive price changes in other pairings. Traders in other markets observe the order flow just transacted and revise their valuations accordingly. Danielsson et al. (2012) stated that there was clear evidence of order flow information being transmitted across linked, triangulated exchange rate markets. Triangulation communicates information on the impact of bid-ask spreads and a mechanism to dissipate order flow information ensuring the equilibrium pricing is restored.

Triangulation is the method by which the information flow concerning price, bid-ask spread and order flow is dispersed through the FX market.

Discreteness

Prices move in discrete units, the exact size of which is either determined by regulators, the exchange or simply market convention. Discreteness induces negative serial correlation in high frequency data: Harris (1990). This occurs because discreteness causes the high frequency data to overstate volatility. Price discreteness impacts the bid-ask spread due to the minimum tick or pip size which establishes the lower bound on the spread, and the size of the increment by which it can be increased. In its own right price discreteness is important in shaping bid-ask spreads and price volatility. However McGroarty et al. (2007) highlighted the fact that the bid-ask spreads are not uniform across the whole of the electronic interdealer spot FX market. If certain prices are used more often than others: price clustering, the effect due to price discreteness will be compounded.

Clustering

Transactions between FX banks and their customers are bilateral and are not visible to other banks. Informed traders take an inventory from their customers, giving rise to inventory imbalances. Banks off-load that inventory near the mid-market rates. These transactions set the mid-price that banks use when quoting the bid-ask spread spot prices to their customers. Price clustering: the distribution of trading volumes across the final digit of trade prices, is an independent factor of the bid-ask spread. In essence price clustering exerts its influence on the bid-ask spreads and volatility.

Why does clustering arise?

Grossman, Miller, Cone, Fischel, and Ross (1997) noted that the lesser degree of clustering on NYSE is the normal result of competition, and the higher degree of clustering on the NASDAQ is anomalous, indicative of a collusive marketplace. The most contentious point was raised by W. G. Christie and Schultz (1994) who suggested that market makers colluded to maintain wider bid-ask spreads by avoiding the odd-eighth digit quotes. W. G. Christie and Schultz (1994) claimed that the multiple dealer market is designed to produce narrow bid-ask spreads through the competition of order flow. They found that odd-eighth quotes were virtually non-existent for 70 out of 100 actively traded NASDAQ securities, implying that spreads are actually twice as wide and maintained by tacit collusion.

The collusion hypothesis as an explanation for clustering is still open for debate. Other plausible reasons include Yule et al. (1927), simply arising systematically from rounding 'human error' when ask to read a numbered scale.

Goodhart and Curcio (1991) put forward a comparison of the *price attraction* and the Ball, Torous, and Tschoegl (1985) *price resolution* hypotheses as possible explanations. The attraction hypothesis focuses on the rounding behaviour of 'human error' identified above. The resolution hypothesis contends with clustering as a natural occurrence in the market. Clustering is the consequence of the compromise between increased price accuracy and the inconvenience of ever longer prices.

Another explanation is the *negotiation* hypothesis suggested by Harris (1991) and Brown, Laux, and Schachter (1991). The idea of a two-tier pricing system whereby large trades result in harder bargaining and more finely tuned prices, while small trades attract rudimentary pricing from a reduced selection of prices. Grossman et al. (1997) identified a variety of factors that potentially contribute to clustering: such as market structure or whether the quotes are binding or not, and found they do not contradict the final-digit price clustering pattern associated with price resolution, rather they attempt to explain why it occurs at all.

3.3 Data Set and Methodology

For the first time since 2001, global FX trading between two consecutive Bank for International Settlements (BIS) surveys: 2013 and 2016, had declined, to recover in 2019. The BIS survey reported that global FX turnover fell to $5.1 trillion per day in April 2016, from $5.4 trilling in April 2013 and recovered to $6.6 trillion in April 2019. Spot trading increased to $1.99 trillion per day in April 2019, from $1.65 trillion in 2016. *See* Table 3.2 for a breakdown of the percentage market share of the leading spot rates as per the *BIS Surveys of 2016 and 2019*.

TABLE 3.2: BIS Survey, April 2019: Market Share of Spot Transactions by Currency Pair

Exch. Rate	April 2013	April 2016	April 2019
EUR/GBP	1.90%	2.00%	2.00%
EUR/USD	24.10%	23.10%	24.00%
GBP/USD	8.80%	9.30%	9.60%
USD/JPY	18.30%	17.80%	13.20%
EUR/JPY	2.80%	1.60%	1.70%
GBP/JPY	1.70%	0.90%	1.00%

Source: BIS Survey, April 2016 and 2019.

See Table 3.3 for the percentage share of average daily turnover for the top four currencies by market share. Because there are two currencies involved in a currency pair, percentage share of individual currencies total is out of 200% instead of 100%.

TABLE 3.3: Currency distribution of OTC FX average daily turnover

Currency	April 2013	April 2016	April 2019
USD	87.00%	87.60%	88.30%
EUR	31.40%	31.40%	32.30%
JPY	23.00%	21.60%	16.80%
GBP	11.80%	12.80%	12.80%

Source: BIS Survey, April 2016 and 2019.

FX trading continued to be dominated by *other financial institutions*: including non-reporting banks, institutional investors and hedge funds, accounting for 51% of turnover in April 2016 increasing to 55% of turnover in April 2019. *Reporting dealers*: executing trades and managing risks on their accounts and on behalf of their clients, accounting for 42% in April 2016 only accounted for 38% of turnover in April 2019 and *non-financial customers*: FX transactions associated with global trade, remained unchanged at 7% in April 2019. However, the share of trading between reporting dealers has increased only once since 1995 in April 2016 to fall back again in April 2019. Interdealer trading, which averaged $2.1 trillion in April 2016, accounting for 42% of turnover, increased in absolute terms to $2.5 trillion but declined relative to the market share to 38% of FX turnover in April 2019. The rise in interdealer trading was primarily driven by the increased trading in FX swaps, a 13% rise since 2013 and 2% rise since 2016 to $3.3 trillion in April 2019. Turnover in spot activity among reporting dealers recovered in absolute terms to $1.99 trillion in April 2019 compared to $1.65 trillion in April 2016.

The Refinitiv Eikon trading system, formerly Thomson-Reuters, provided the data for this study. The data set covers the three month period from September 12th, 2018, to December 12th, 2018. The data consists of over ninety thousand observations for each of the top four currencies by percentage share of average daily turnover as per the April 2019 BIS survey, *see* Table 3.3. The related currency pairs consists of EUR/USD, EUR/GBP, GBP/USD, USD/JPY, EUR/JPY and GBP/JPY, *see* Table 3.2.

The exchange rate price history comprises the best bid, mid and ask quotes and time-stamped to the nearest second. No information as to the transaction size or trading parties is given.

The analysis looks at the impact on exchange rates due to *bid-ask spreads* and reveals the price discreteness properties of each rate. The term *pip* is commonly used in the foreign exchange market in place of the word *tick*. It may be worth acknowledging the distinction that pips arise as a matter of convention, whereas ticks are formally enforced, usually by an exchange, in this paper the use of the two terms is inter-changeable. A pip usually refers to the incremental value in the fifth non-zero digit position from the left. Note that it is not related to the position of the decimal point. For example, one pip in a USD/JPY value of 113.57 would be 0.01, while one pip for EUR/USD of 1.0434 would be 0.0001. The fact that the decimal place does not occupy a fixed position necessitates the introduction of a *scaling factor* with the purpose of bringing the pip to the left of the decimal point. For example, the scaling factor for the USD/JPY is 100 and that for the EUR/USD is 10,000.

The pip bid-ask spread is defined as

$$pip = (ask - bid) \cdot Scaling\ Factor\,. \tag{3.1}$$

The first part of the empirical analysis considers the price discreteness properties of each exchange rate by computing the summary statistics of the pip bid-ask spread.

The next part looks at establishing the presence of price clustering by applying a diagnostic chi-squared, χ^2, goodness-of-fit test statistic to the observed set of final digits.

McGroarty et al. (2006) proposed the χ^2 test for clustering. If clustering is absent then an equal number of observations for each digit from 0 to 9 will give a χ^2 value of zero. However there is a limitation to the χ^2 test which is very sensitive to sample size. With a large enough sample, even trivial relationships can appear to be statistically significant. For χ^2, larger samples may lead to the decision of rejecting the null hypothesis, concluding the presence of clustering in error. In interpreting the χ^2 goodness-of-fit for the large sample size undertaken in this test, consider whether the interpreted test statistic is meaningful. The χ^2 statistic is

$$\chi^2 = \sum_{i=1}^{10} \frac{\left(n_i - \frac{1}{10}\sum\limits_{j=1}^{10} n_j\right)^2}{\frac{1}{10}\sum\limits_{j=1}^{10} n_j}\,, \tag{3.2}$$

where $n_{i(j)}$ equals the number of observations at final digit $i(j)$. The χ^2 critical value at the 1% significance level with 9 degrees of freedom is 21.666. The null hypothesis test is for no significant difference between the observed and expected values: the absence of clustering. The χ^2 statistic only addresses the existence of price clustering, it does not determine its level.

The extent of clustering can be summarised numerically by measuring the range between the highest and lowest quotation frequencies relative to the expected final digit distribution with no clustering. To enable comparison between markets Grossman et al. (1997) proposed the Standardised Range

(SR) test as a measure of the level of clustering. The *SR* formula is given as

$$SR = (Hi(\phi_i') - Lo(\phi_i'))/Q \,, \tag{3.3}$$

where ϕ_i' is the percentage of final digit observations as a proportion of the total population and $(Hi(\phi_i') - Lo(\phi_i'))$ is the difference between the highest and lowest percentage frequency at digit i: the maximum and minimum percentage distribution of last digit quote, and $Q = 10\%$: the percentage at each final digit i, if no clustering occurs. The absence of clustering leads to an *SR* value of zero and a 100% concentration would give a value of ten.

The χ^2 and *SR* tests address the existence and magnitude of price clustering but reveal nothing of the cause. Goodhart and Curcio (1991) provide two ordered final digit groupings that correspond to the two possible causes of clustering discussed previously:

i. If the *attraction hypothesis* is correct the final-digits should occur in descending order of frequency with 0 & 5 the most frequent, followed by the pairs 7 & 3, 8 & 2, 4 & 6 and the least frequent pair 1 & 9. The test statistic is

$$A = \frac{Min((Av(n_{3,7}) - Av(n_{2,8})), (Av(n_{2,8}) - Av(n_{4,6})), (Av(n_{4,6}) - Av(n_{1,9})))}{Av(|n_3 - n_7|, |n_2 - n_8|, |n_4 - n_6|, |n_1 - n_9|)} \,, \tag{3.4}$$

where n_i is the number of observations at the final digit i, n_{ij} is the set of numbers of observations at the final digits i and j, Min is the minimum value of the set and Av is the mean value with $||$ being the absolute value. The numerator is the minimum difference between the grouped average set of observations. The denominator is the average absolute difference between the same grouped observations.

ii. If the *resolution hypothesis* is correct the final-digits should occur in descending order of frequency with 0 & 5 the most frequent, followed by two groups with 2 & 3 & 7 & 8 in one group and 1 & 4 & 6 & 9 in the least frequent group. The test statistic is

$$R = \frac{Av(n_{2,3,7,8}) - Av(n_{1,4,6,9})}{Max(Max(n_{2,3,7,8}) - Min(n_{2,3,7,8}), Max(n_{1,4,6,9}) - Min(n_{1,4,6,9}))} \,, \tag{3.5}$$

where n_i is the number of observations at the final digit i, $n_{i,j,k,l}$ is the set of numbers of observations at the final digits i, j, k and l, Max and Min are the maximum and minimum value of the set respectively and Av is the mean value. The numerator is the difference between average grouped set of observations. The denominator is the maximum value of the difference between the maximum and minimum value of each group set.

Equations (3.4) and (3.5) depend on 0 and 5 being the two most frequent end digits. If they are not, then the pattern does not comply with the conventional forms being tested and the test value is set to zero. Otherwise, the numerator of Equation (3.4) takes the average of the observations for each sequential pair and then detects the minimum difference between each of these averaged pairs. The denominator calculates the average of the absolute difference within each ordered pair. The smallest difference between the pairs is divided by the average difference within the pairs. Note that the denominator will always be non-negative, so that a negative number on the numerator can never be made positive by the denominator. For both tests a test statistic greater than one denotes strong evidence in favour of the respective hypothesis. Positive values below one suggest the right ordering but a weak fit, where the higher a positive number the better the fit and a non-positive value rejects the hypothesis.

3.4 Results

The empirical results present price granularity as a proxy for price discreetness, Goodhart, Love, Payne, and Rime (2002). Table 3.4 presents the summary statistics for the pip spot FX data to evaluate price discreteness as a source of wider bid-ask spreads.

TABLE 3.4: Currency Pairs - Pip Spread Statistics

	EUR/GBP	EUR/USD	GBP/USD	USD/JPY	EUR/JPY	GBP/JPY
Mean	3.462	2.731	3.179	2.261	3.701	3.785
Median	4.000	3.000	4.000	2.000	4.000	5.000
Maximum	16.000	11.000	23.000	8.000	8.000	12.000
Minimum	1.000	1.000	1.000	0.900	1.000	1.000
Std. Dev.	1.507	1.365	1.554	1.054	1.386	1.661
Skewness	−0.371	0.166	−0.131	0.223	−0.977	−0.589
Kurtosis	2.556	2.054	2.323	2.299	2.813	2.035
Jarque-Bera	2945.287	4004.775	2125.109	2767.601	15 287.220	9169.680
p-value	0.000	0.000	0.000	0.000	0.000	0.000
Sum	327 917.000	261 570.000	307 840.000	217 833.700	352 255.000	359 034.000
Sum Sq. Dev.	214 985.100	178 387.700	233 705.600	106 969.600	182 936.800	261 703.100
Observations	94,709	95,763	96,838	96,333	95,171	94,850

Eviews 9

The first part of the analysis in Table 3.4 reveals the price discreteness properties of the pip bid-ask spread. The low standard deviation shows that the spreads are grouped close together around the mean. The mean and median are not vastly different, indicating the absence of excess outliers. Note that the mean is less than the median for all currency pairs except USD/JPY, implying the data is negatively skewed to the left: long left tails containing more spreads below the mean than above the mean. Conversely the USD/JPY currency pair is positively skewed: long right tails with more data above the mean than below it. This is in line with the skewness figures obtained. With kurtosis of less than 3 the distributions are *platykurtic* producing fewer and less extreme outliers than would be expected with a normal distribution. The large Jarque-Bera statistic rejects the null hypothesis of normality concluding the pip bid-ask spread is not normally distributed across all the exchange rates considered.

Goodhart et al. (2002) considered whether the market practice relating to pip quoting could have been changed to add a decimal place and so facilitate a reduction in bid-ask spreads: traders could use six rather than five significant digits. They concluded that this could have introduced complications as other researchers found that trade size and market depth fell when smaller pip sizes were imposed. The result support the view that price discreteness sets the parameters for the spreads, the majority of spread data being closely grouped and less than the mean with very few outliers, but does not indicate that increasing the pip to six significant digits from five would alter the distribution of the spread once the scaling factor has been applied to compensate for the additional digit. Adding an additional digit only increases the scaling factor not the distribution. Although there is no central regulatory body, the interdealer spot FX market seems to have accounted for price discreteness in the exchange rates by fully utilising the final digit pip quotes. Price discreteness is factored into the FX pricing structure and market mechanism.

In considering if price clustering behaviour may explain the action of bid-ask spreads the final digit distribution is applied to all currency pairs and the W. G. Christie and Schultz (1994) collusion hypothesis is tested by examining the odd versus even quote distribution. The currencies analysed in this study consist of five significant digits throughout, as is the convention in the spot FX market, and all the rates use the full final digit range of 0 to 9. Tables 3.5 and 3.6 present summary statistics relating to price clustering within the quoted price data from submitted limit orders.

TABLE 3.5: Currency Pairs - Percentage Distribution of Spot FX Quotes at Final Digit

	Distribution of Last Digit Quote									
Exch. Rate	0	1	2	3	4	5	6	7	8	9
EUR/GBP	10.30%	9.75%	9.88%	9.50%	9.63%	9.97%	9.88%	10.13%	10.22%	10.75%
EUR/USD	9.48%	9.06%	9.87%	10.31%	10.58%	10.67%	10.25%	10.26%	10.07%	9.45%
GBP/USD	10.00%	9.77%	10.02%	9.88%	10.05%	9.94%	9.94%	10.08%	10.29%	10.03%
USD/JPY	10.40%	9.47%	9.89%	10.02%	9.69%	9.78%	9.79%	10.25%	10.20%	10.50%
EUR/JPY	9.95%	9.97%	9.57%	9.88%	10.06%	10.26%	10.27%	10.07%	10.02%	9.97%
GBP/JPY	10.04%	9.56%	9.82%	10.22%	9.91%	10.21%	10.22%	9.68%	10.34%	9.99%

TABLE 3.6: Currency Pairs - Odd and Even Digit Distribution

Exch. Rate	Odd Digits	Even Digits
EUR/GBP	50.1 %	49.9 %
EUR/USD	49.7 %	50.3 %
GBP/USD	49.7 %	50.3 %
USD/JPY	50.0 %	50.0 %
EUR/JPY	50.1 %	49.9 %
GBP/JPY	49.7 %	50.3 %

Table 3.5 presents the percentage distribution of spot FX quotes at the corresponding final digit. By inspection one can see that the final digits 0 and 5 are not significantly dominant over the remaining final digits as suggested by Goodhart and Curcio (1991). However the distribution across the number range is not an even 10%, suggesting clustering is potentially present. The null hypothesis of the absence of clustering will be tested below, *see* Table 3.7. Table 3.6 presents the odd versus even number usage among full price points in spot FX markets. The usage demonstrates that there is no clear evidence that odd numbers are preferred to even, or *vice versa*, in the electronic interdealer spot FX market for the currencies considered. This contrasts with the McGroarty et al. (2006) test which showed persistent evidence that even numbers were weakly preferred to odd numbers.

TABLE 3.7: Currency Pairs: χ^2 Test of Price Clustering

Exch. Rate	No. of Obs.	χ^2-test	p-value	Null
EUR/GBP	94,709	113.707	0.0000 %	*Reject*
EUR/USD	95,763	237.962	0.0000 %	*Reject*
GBP/USD	96,838	16.741	5.2931 %	*Accept*
USD/JPY	96,333	96.574	0.0000 %	*Reject*
EUR/JPY	95,171	33.483	0.0110 %	*Reject*
GBP/JPY	94,850	55.998	0.0000 %	*Reject*

χ^2 Test Statistic. Critical value at the 1% significance level with 9 degrees of freedom is 21.666.

If no clustering is present then the distribution across the range 0 to 9 of final digits in Table 3.5 should be 10% each. This is not the case and a χ^2 test is undertaken, Equation (3.2), to verify the null hypothesis of no significant difference between the observed and expected values in the spread distribution: no clustering. Table 3.7 presents the χ^2 test statistic for the presence of price clustering in the spot FX market. The p-value represents the percentage probability that the difference between observed and expected outcomes stem from sampling error. Therefore if p-value is less than 0.01 (1% level of significance) the null hypothesis is rejected and conclude clustering is present. The χ^2 critical value at 9 degrees of freedom is 21.666, the test statistic is greater than the critical value for all currency pairs with the exception of GBP/USD and thus reject the null and conclude that clustering exists in all currencies tested except GBP/USD.

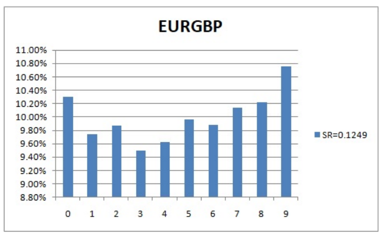
Whereas the χ^2 test confirms the presence of clustering, Grossman et al. (1997) proposed the Standardised Range *(SR)* test, Equation (3.3), as a measure of the level of clustering to enable comparison between markets. The absence of clustering would lead to a value of zero, whereas 100% concentration would give an *SR* value of ten. *see* Table 3.8.

TABLE 3.8: Currency Pairs - Standardised Range Test

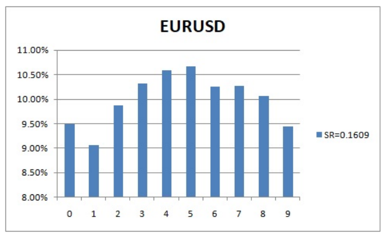
Exch. Rate	Standardised Range
EUR/GBP	0.1249
EUR/USD	0.1609
GBP/USD	0.0528
USD/JPY	0.1033
EUR/JPY	0.0701
GBP/JPY	0.0778

Standardised Range Test Statistic. No clustering returns a value of zero. 100% clustering returns a value of ten.

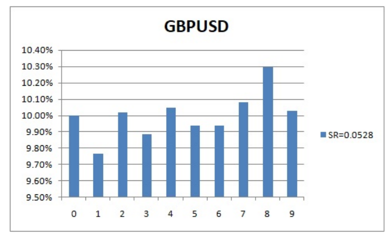
FIGURE 3.1: Currency Pairs - Standardised Range Graphs



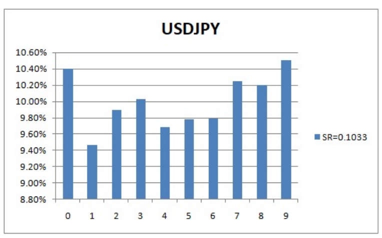
(a) EUR/GBP Graph 1



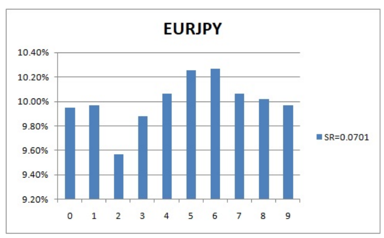
(b) EUR/USD Graph 2



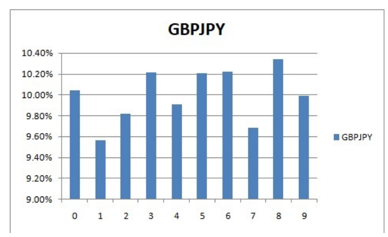
(c) GBP/USD Graph 3



(d) USD/JPY Graph 4



(e) EUR/JPY Graph 5



(f) GBP/JPY Graph 6

The Standardised Range measures in Table 3.8 and Figure 3.1 show low levels of price clustering in spot FX rates. The Standardised Range test shows that the EUR/USD, the most traded currency pair with 24.0% of the market, has the most clustering and GBP/USD, with only 9.6% of the market the least. This corresponds to the χ^2 test in Table 3.7 where GBP/USD was the only currency pair that did not reject the null hypothesis of no clustering. The low degree of clustering for EUR/JPY, with 1.7% of the market, is what is expected as the EUR/JPY exchange rate is less volatile than the EUR/USD, EUR/GBP and USD/JPY exchange rates which collectively make up 39.2% of the FX market.

Having confirmed the presence and magnitude of clustering in the FX market, now consider the causes of price clustering behaviour that may explain the action of bid-ask spreads. The Goodhart and Curcio (1991) price clustering pattern test is applied to all currency pairs and reveals the price clustering patterns in quoted spot FX limit orders. The test proposed by Goodhart and Curcio (1991) evaluates the attraction and resolution hypotheses as an explanation of the possible causes of clustering.

TABLE 3.9: Currency Pairs - Attractions & Resolution Price Clustering Pattern Test

Exch. Rate	Attraction[a]	Test	Resolution[b]	Test
EUR/GBP	−0.8870	*Reject*	−0.0613	*Reject*
EUR/USD	−1.8260	*Reject*	0.1931	*WeakFit*
GBP/USD	0.0000	*Reject*	0.0000	*Reject*
USD/JPY	−0.5872	*Reject*	0.2219	*WeakFit*
EUR/JPY	−1.7706	*Reject*	−0.3700	*Reject*
GBP/JPY	−0.2909	*Reject*	0.1386	*WeakFit*

[a] Attraction Clustering Pattern: Test statistic greater than one denotes strong evidence on favour of the premiss. Positive values less than one suggested the right ordering but a weak fit. Non-positive values reject the premiss.
[b] Resolution Clustering Pattern: Test statistic greater than one denotes strong evidence on favour of the premiss. Positive values less than one suggested the right ordering but a weak fit. Non-positive values reject the premiss.

The initial test is to verify whether 0 and 5 are the two most frequent final digits in Equations (3.4) and (3.5). If they are not, then the pattern does not adhere to the conventional forms being tested and the test value is set to zero. In testing the premiss a test statistic greater than one denotes strong evidence in favour of the proposition. A positive number shows increasing levels of acceptance. The higher the value, the better is the fit. Positive values below one suggest the right ordering but a weak fit. Positive values below one identify series with the appropriate rank ordering but differences within at least one of the groupings dominates the difference between the groups. Non-positive test values reject the premiss, denoting that the sets do not exhibit the ordering required by the test.

Table 3.9 reveals the results of the attraction and resolution price clustering tests. The tests look to explain the frequency of final digit usage in quotes for the FX spot price data. In terms of the formal test for the cause of price clustering, the attraction and resolution price pattern tests for GBP/USD, *see* Table 3.9, were rejected: digits 0 and 5 did not conform to the patterns being tested and the test statistic values were set to zero, rejecting the premiss as an explanation for clustering for this currency pair. This is consistent with the χ^2 and *SR* tests above.

The detailed breakdown for the remaining currency pairs showed a slight favour of the final digits 0 and 5, enabling the attraction and resolution test for the cause of price clustering. Regarding the attraction price pattern, *see Attraction* Table 3.9, the remaining currency pairs were all non-positive and the proposal rejected. Concluding that the attraction hypothesis does not explain the clustering patterns in the currency pairs considered.

For the resolution price pattern, *see Resolution* Table 3.9, currency pairs EUR/GBP and GBP/JPY were non-positive and the proposal rejected. EUR/USD, USD/JPY and GBP/JPY, the three most traded currency pairs, were slightly positive and weakly fit, accepting the premiss put forward by Ball et al. (1985) that clustering in these markets is a natural occurrence in a market which has reached the optimal degree of price resolution.

3.5 Conclusions

This chapter examined the microstructure approach to exchange rates: order flow and the bid-ask spread, and the corresponding relationship with triangulated pricing in seeking to identify the determining factors that model the spot price in endeavouring to explain the function of the FX markets.

The explanatory microstructure factors of the bid-ask spread *i.e.* price discreteness and price clustering were appraised by gauging the changes in the spreads for six currency pairs that account for 51.5% of the spot FX market by volume to identify the pricing characteristics of the market. An additional test proposed by Goodhart and Curcio (1991) was applied to address the pattern of clustering in an attempt to identify the potential causes.

Appraising the price discreteness found no indication that increasing the spot FX quotes from five to six digits would decrease the spread. Rejecting the proposal of an additional digit to the spot price quotes as a means of enhancing the transmission mechanism and concluding that the transfer of market information is efficient, the extra digit would simply increase the required scaling factor and not affect the distribution. The distribution of the price data was closely grouped around the mean for almost all the currencies tested, with a small standard deviation implying that the quoted price is in the region of the market average with only a small margin of error. Hence, price discreteness has been efficiently assimilated into the trading mechanism.

The presence of price clustering was evaluated by applying the χ^2 test statistic rejecting the null hypothesis of no significant difference between the observed and expected values in the spread distribution, concluding that clustering was present. There was obvious price clustering present in the majority of the currencies in the spot FX markets examined. However, applying the Standardised Range *SR* test to measure the level of clustering revealed that where clustering was present it was not significant, returning very low values in the *SR* test.

The spread pattern was then appraised to ascertain the possible causes of clustering. The price attraction hypothesis was overwhelmingly rejected as an explanation for price clustering behaviour in the spot FX market and only weakly accepted the price resolution explanation. Both price discreteness and price clustering, although present, are incorporated into an efficient market and factored into the pricing processes that make up the information transmission mechanism.

Concluding that order flow is a mechanism for dissipating net buying pressure, the bid-ask spread manifests the transaction costs and associated risks whereas triangulation is the vehicle used to restore the market equilibrium for the microstructure attributes and price, the proposed approach to model exchange rates as a system of attributes is credible. The model describes the efficient information flow through the FX market that determines the spot price. These attributes are essential to explain how an efficient market functions and are fundamental to defining the model behaviour that can be used as the basis of an FX option pricing formula. The roles of order flow, bid-ask spreads and triangulation are essential factors that must be taken into consideration, in concert, when developing a theoretical stochastic model of the FX market.

Chapter 4

Stochastic Stock Price Processes

This chapter considers the theoretical models underpinning the stochastic stock equations that define the Black and Scholes (1973) and R. C. Merton (1973) (BSM) option pricing formula. When viewing the stochastic stock processes in relation to foreign exchange a specific characteristic of the transaction has to be taken into consideration: arbitrage. Equity transactions in one sector may result in an investor having to reallocate some of their equity portfolio holdings, which could have an indirect effect on the pricing of equity in other sectors. However the interactions between exchange rates are stronger because of arbitrage. Bilateral foreign exchange rates, quoted in currency pairs, are directly affected by the trilateral pricing movement of related triangular pairings due to arbitrage. In this chapter the stochastic stock price process that predicates the assumption that security prices follow a geometric Brownian motion (GBM) is examined in considering how the process can be applied to foreign exchange. The purpose of this chapter is to emphasize the unique differences of the stochastic stock equations with respect to foreign exchange and review the relevant literature as the basis for extending the stock models to define the stochastic processes that describe the foreign exchange market.

4.1 The Properties of Price Processes

The properties of the stochastic price processes relating to Brownian motion and geometric Brownian motion are presented in Appendix B on page 152.

4.1.1 The Validity of the BSM Geometric Brownian Motion Assumption

Why choose geometric Brownian motion?

Geometric Brownian motion (GBM) is frequently invoked as the model for stock price returns. It is widely accepted as a valid model for the growth in the prices of stocks over time and forms the basis for the Black and Scholes (1973) option pricing model. It is important to verify that a time series follows a GBM process before relying on the results for such an assumption.

Any stochastic process containing a Wiener process is a normally distributed stochastic process with mean 0, standard deviation $\sqrt{t}$ and variance t.

Geometric Brownian motion replaces the generalised Wiener process constant drift rate a with the expected rate of return μ and the noise multiplier b with the variability of the percentage return in a short period of time, regardless of the stock price, thus the standard deviation is proportional to the stock price σS, *see* Section B.1.3 on page 155. The model of stock price behaviour described by geometric Brownian motion is therefore given as

$$\frac{dS}{S} = \mu\, dt + \sigma\, dz\,. \tag{4.1}$$

From Itô's lemma, the process followed by $\ln S$ when S follows the process in Equation (4.1) is given as

$$\ln S_T - \ln S_0 \sim \phi\left[(\mu - \frac{\sigma^2}{2})T, \sigma^2 T\right]\,, \tag{4.2}$$

where S_T is the stock price at a future time T and S_0 is the stock price at time 0.

Black and Scholes (1973) concluded that stock price returns described by geometric Brownian motion are normally distributed $\sim N\left(\mu dt, \sigma^2 t\right)$ with mean μdt and variance $\sigma^2 t$. If the logarithmic return $\frac{dS}{S}$ is normally distributed as $\sim N\left[\left(\mu - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right]$ then the logarithmic prices $\ln S_T$ are normally distributed with characteristic function $\sim \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right]$ as per the Black and Scholes (1973) assumptions on page 194.

Although geometric Brownian motion proved a better fit to the distribution of actual stock price changes than Brownian motion, Malone (2002) posits that geometric Brownian motion is not the most accurate model that can be used for the evolution of stock price processes. Other diffusion processes may be used to model price evolution, significantly sample paths may not be continuous and jump processes may be considered in addressing any discrepancies.

4.1.2 Diffusions, Jump Processes and Mixed Jump Diffusion Processes

In presenting potential processes to replace geometric Brownian motion to model the evolution of prices in time, alternative diffusions, jump processes and mixed jump diffusion processes are considered. Given that the stochastic process literature is so large it is essential to be selective in this review. Therefore this review covers models that are used for related applications in finance.

The Constant Elasticity of Variance Model

Cox and Ross (1976) suggested a class of variance, $\sigma(S,t)$, that is referred to as the constant elasticity of variance model. The differential equation for the stock price S becomes

$$dS = \mu S\, dt + \sigma S^{1-\alpha}\, dz\,, \tag{4.3}$$

with $\sigma(S,t) = \sigma S^{1-\alpha}$ for some α where $0 \leq \alpha \leq 1$. Setting $\alpha = 0$ gives geometric Brownian motion. Two processes of note which are special cases of the constant elasticity of variance model are:

i. Linear Price Variance Process:
$dS = \mu S\,dt + \sigma\sqrt{S}\,dz$

ii. Constant Price Variance Process:
$dS = \mu S\,dt + \sigma\,dz$,

The Linear Price Variance Process describes a situation where changes are small and the variance of price changes increases with the stock price but at a slower rate than in geometric Brownian motion. Thus for the Linear Price Variance Process the variance of the rate of return decreases rather than remain constant. Conversely the Constant Price Variance Process characterises stock price changes that have a constant variance. Both the Linear Price Variance and Constant Price Variance Processes are diffusion limits of jump processes highlighted by Cox and Ross (1976).

Poisson Processes

A Poisson process is a simple and widely used stochastic process for modelling the times at which arrivals enter a system. Arrivals may occur at arbitrary positive times, and the probability of an arrival at any particular instant is 0. It is convenient to define a Poisson process in terms of the sequence of interarrival times: times between successive arrivals, $t_1, t_2, \ldots$, which are defined to be independent and identically distributed (i.i.d.).

A Poisson process is a renewal process: arrival process for which the sequence of arrival times is an i.i.d. random variable $X'(t)$, in which the interarrival intervals have an exponential distribution function for some real $\lambda \geq 0$, and each $X'(t)$ has the density

$$f(X'(t)) = \lambda e^{(-\lambda t)} \quad \text{for} \quad t \geq 0\,. \tag{4.4}$$

The parameter λ is the arrival rate of the process. For any interval of size t, λt is the expected number of arrivals in that interval.

A random variable $X'(t)$ taking values in $\{0, 1a, 2a, 3a, \ldots\}$ in k events is said to have a Poisson distribution with parameter $\lambda \geq 0$ and jump size $a \geq 0$ if

$$P(X'(t) = ka) = \frac{(\lambda t)^k}{k!} e^{-\lambda t} : k \in (0, \mathbb{I}^+)\,. \tag{4.5}$$

A process $X'(t)$ with stationary, independent increments has a version with all sample paths constant except for upward jumps of length one if and only if there is a parameter $\lambda \geq 0$ such that the characteristic function takes the form

$$\phi(X'(t)) = e^{[\lambda(e^{it}-1)]}\,. \tag{4.6}$$

The Poisson process is a process with stationary, independent increments. Brownian motion shares these properties with the Poisson process, but unlike Brownian motion, the Poisson process is discontinuous, its sample paths are constant except for upward jumps.

Jump Processes

Cox and Ross (1976) noted that the jump process represents the movement of stock prices associated with information arriving in packages rather than as a continuous stream. If x denotes the current state of the world, then the Markov jump process takes the form

$$dS = \mu(x)\,dt + dq\,, \tag{4.7}$$

where dq are the increments of the pure jump process q, given by

$$dq = \begin{cases} k(x) - 1, & \text{with probability } \lambda(x)\,dt\,, \\ 0, & \text{with probability } 1 - \lambda(x)\,dt\,, \end{cases} \tag{4.8}$$

where $k(x)$ has a distribution dependent on the current state x, and λ is the intensity of the process: the average number of events per interval.

Cox and Ross (1976) assumed the current state $x = S$ to reflect that all states of information are contained in the current stock price S.

To demonstrate the particular jump processes that are special cases of Equation (4.7), that in the limit become the Linear Price Variance and Constant Price Variance processes let the intensity $\lambda(S) = \lambda S$ and the drift $\mu(S) = \mu S$. For the rate of return, dS/S, the drift is constant and assumes the information arrives more frequently when the stock price is higher. Additionally the distribution for the jump in prices is independent of the price. The distribution of the jump component is given by

$$dq = \begin{cases} k - 1, & \text{with probability } \lambda S\,dt\,, \\ 0, & \text{with probability } 1 - \lambda S\,dt\,. \end{cases} \tag{4.9}$$

Hence the stochastic process for price changes becomes

$$dS = \mu S\,dt + dq\,. \tag{4.10}$$

Cox and Ross (1976) described Equation (4.10) as a generalisation of a class of stochastic processes known as the birth and death process: *births* increase the state variable by one and conversely *deaths* decrease the state variable by one. The local mean and variance for Equation (4.10) are given by

$$\begin{aligned} & E\,[dS] = (\mu + \lambda\,E\,[k-1])S\,dt \\ \text{and} & \\ & Var\,[dS] = \lambda\,E\left[(k-1)^2\right] S\,dt\,. \end{aligned} \tag{4.11}$$

To construct a pure birth and death process, ignore the drift in Equation (4.10) and let the random variable k take on two values, $k^+ > 1$ and $k^- < 1$ with respective conditional probabilities π^+ and π^-. This gives the stochastic process

$$dS = \begin{cases} k^+ - 1, & \text{with probability } \pi^+ \lambda S\, dt\,, \\ k^- - 1, & \text{with probability } \pi^- \lambda S\, dt\,, \\ 0, & \text{with probability } 1 - \lambda S\, dt\,. \end{cases} \tag{4.12}$$

If a stock price S is made up of individual, stochastically independent units whose sum value is S, then λdt represents the probability of an event occurring for any one unit. An event is considered a birth process if $k^+ - 1$ additional units are added with probability π^+, and conversely a death process event has $1 - k^-$ fewer units with probability π^-.

Taking the limits of Equation (4.10) as $k^+ \longrightarrow 1$ and $k^- \longrightarrow 1$ and $\lambda \longrightarrow \infty$ produces the Linear Price Variance Process

$$dS = \mu S\, dt + \sigma \sqrt{S}\, dz\,, \tag{4.13}$$

were μ and σ are given by

$$\begin{aligned} & \mu = \lambda E[k-1] \\ \text{and} \qquad & \\ & \sigma = \sqrt{\lambda E[(k-1)^2]}\,. \end{aligned} \tag{4.14}$$

Another specialisation of the general Markov jump process, Equation (4.7), can be used to obtain the Constant Price Variance Process. To accomplish this assume the intensity λ and the value increments are both constant. Then the jump component for the stochastic process given by Equation (4.10) becomes

$$dq = \begin{cases} k^+ - 1, & \text{with probability } \pi^+ \lambda\, dt\,, \\ k^- - 1, & \text{with probability } \pi^- \lambda\, dt\,, \\ 0, & \text{with probability } 1 - \lambda\, dt\,. \end{cases} \tag{4.15}$$

Cox and Ross (1976) refers to this as the absolute process where values grow endogenously at the exponential rate μ and where bulk exogenous changes to the value of size $k - 1$ occur with intensity λ. The local mean and variance of the absolute process are given by

$$\begin{aligned} & E\,[dS] = \left(\mu S + \lambda \left[\pi^+(k^+ - 1) + \pi^-(k^- - 1)\right]\right)\, dt \\ \text{and} \qquad & \\ & Var\,[dS] = \lambda \left[\pi^+(k^+ - 1)^2 + \pi^-(k^- - 1)^2\right]\, dt\,. \end{aligned} \tag{4.16}$$

Taking the diffusion limit of the absolute process of Equation (4.10) yields the diffusion for the Constant Price Variance Process

$$dS = \mu S\, dt + \bar{\sigma}\, dz\,, \tag{4.17}$$

where the drift μ is the same as the drift in the absolute process and the standard deviation $\bar{\sigma}$ is given by

$$\bar{\sigma} = \sqrt{\lambda\, [\pi^{+}(k^{+}-1)^2 + \pi^{-}(k^{-}-1)^2]}\,. \tag{4.18}$$

In taking the limit of the jump process to obtain the diffusion, whilst maintaining the instantaneous mean and variance, the mean of the jump process is set to zero so that the resulting Wiener process dz has mean zero.

Now consider which jump process has geometric Brownian motion as its limit? The answer is another modified general Markov jump process from Equation (4.7) given as

$$dS = \mu S\, dt + dq\,, \tag{4.19}$$

with jump component

$$dq = \begin{cases} k^{+} - 1, & \text{with probability } \lambda\, dt\,, \\ 0, & \text{with probability } 1 - \lambda\, dt\,, \end{cases} \tag{4.20}$$

where the intensity λ is independent of the stock price S.

For geometric Brownian motion the variance of the price changes is proportional to a constant (σ^2) times the square of the stock price. This limiting behaviour in the process above is due to the units of value comprising the stock price which are dependent on S, whereas the intensity λ is independent of the stock price S. When a new package of information arrives, the units comprising the stock price are affected equally. The dependence of events is a characteristic of the absolute process.

Mixed Jump Diffusion Processes

The difference between the jump processes, which leads to the differences between limiting diffusions, is that in the jump process leading to geometric Brownian motion the jump size is proportional to the stock price S.

R. C. Merton (1976) considered the price of a stock that undergoes jumps by noting a stochastic process with a Wiener component as well as a jump component as

$$\frac{dS}{S} = (\mu - \lambda k)\, dt + \sigma\, dz + dq\,, \tag{4.21}$$

where μ is the instantaneous expected return on the stock, σ^2 is the instantaneous variance of the return conditional on the Poisson event not having occurred, dz is a standard Wiener process and dq is the increment in the pure jump process q, given by

$$dq = \begin{cases} Y-1, & \text{with probability } \lambda\, dt\,, \\ 0, & \text{with probability } 1-\lambda\, dt\,, \end{cases} \tag{4.22}$$

where Y is a random variable and $Y-1$ is an impulse function: zero everywhere but infinitely high at the origin, producing a finite jump in S to SY. The increment in the pure jump process dq and standard Wiener process dz are assumed to be independent, λ is the intensity or mean number of arrivals per unit time and $k = E[Y-1]$ is the expected percentage change in the stock price if the Poisson event occurs.

R. C. Merton (1976) commented that the resulting sample path for $S(t)$ will be continuous most of the time with finite jumps of differing signs and amplitude occurring at discrete points in time.

4.1.3 Truncated Lévy Flight Processes

Mantegna and Stanley (1994) proposed a stochastic process known as the truncated Lévy flight which exhibits a slow convergence to Gaussian behaviour. This process has application to stock price time series. The truncated Lévy flight is constructed by taking the sum of n i.i.d. random variables X_i with finite variance

$$S_n = \sum_{i=1}^{n} X_i\,. \tag{4.23}$$

The Central Limit Theorem states that as $n \longrightarrow \infty$, the random variable S_n will converge to a normal distribution. The rate of convergence is not specified and depends on the distribution of X_i. Suppose the random variables X_i share the same distributions as the random variable X. Then the truncated Lévy flight is characterised by the probability distribution for X

$$T(x) = \begin{cases} 0, & \text{if } x > l\,, \\ c_1 L(x), & \text{if } -l \le x \le l\,, \\ 0, & \text{if } x < -l\,, \end{cases} \tag{4.24}$$

where c_1 is a normalising constant, l is the cut off length and $L(x)$ the symmetric Lévy stable distribution defined by

$$L(x) = \int_0^{\infty} e^{-\gamma * q^{\alpha}} \cos(qx)\, dq\,, \tag{4.25}$$

where α ($0 < \alpha \leq 2$) is the characteristic exponent and $\gamma* > 0$ the scale factor.

For appropriate values of the parameters of the Lévy stable distribution $L(x)$ and the cut off length l the truncated Lévy flight may provide a reasonable empirical fit to daily logarithmic price data.

4.1.4 The Price Process Assumptions

Geometric Brownian motion is the model on which the Black and Scholes (1973) and R. C. Merton (1973) (BSM) equation is based on. Cox and Ross (1976) pointed out that the critical factor in the BSM analysis was the precise description of the stochastic process governing the behaviour of the basic asset. Further, R. C. Merton and Samuelson (1974) and R. C. Merton (1976) were critical of the assumption in the BSM derivation that trading takes place continuously in time and the price dynamics of stock have a continuous sample path with a probability of one. Cox and Ross (1976) and R. C. Merton (1976) provide an examination of the option pricing problem for alternative stochastic processes.

There are two paths to take for relaxing the assumption that stock prices follow a geometric Brownian motion:

i. Specify an alternative stochastic process for the price and use arguments similar to those used by Black and Scholes (1973) to arrive at the appropriate differential equation, which may be solved using the boundary condition given by the option.

ii. Alternatively to specify a stochastic model for the stock price volatility, $\sigma(S, t)$. Heston (1993) offers a model of stochastic volatility that is not based on the Black and Scholes (1973) formula. Heston (1993) assumes the spot returns follow a diffusion process whereby volatility varies with time. The model shows that a higher variance raises the prices of all options in line with the BSM equation. It is important to realise that the implied variance may not equal the variance of the spot returns given by the true process due to the risk premium associated with exposure to volatility changes. This form of risk premium is obtained by arbitrage and consistent with conditional heteroscedasticity in consumption growth and asset returns. The stochastic volatility model therefore can explain some properties of the option prices in terms of the underlying distribution of sport returns. The disadvantage of these models is that they do not usually have closed-form solutions.

Naturally, the paths of changing the price process assumption explicitly and relaxing the assumption of constant volatility are related. Bakshi et al. (1997) provides a study of option pricing issues that are related to the hybrid stochastic models for volatility and returns

Relaxing the price process assumption that stock prices follow a geometric Brownian motion this paper will follow option (i): specifying an alternative stochastic process for the price used by the Black and Scholes (1973) method to arrive at the appropriate differential equation which may be solved by the option boundary conditions. Although there is no guarantee that this procedure will arrive at a closed-form solution.

4.2 Literature Review

4.2.1 Statistical Study of the Distribution of Stock Price Returns

A method of characterising the behaviour of a random variable is to describe it in terms of a distribution function. Officer (1972) reviewed the distribution of stock price returns to verify if the distribution is best described by the symmetric stable class of distributions. A distribution is stable if a linear combination of two independent random variables has the same distribution as the individual variables. Officer (1972) noted that the distribution of stock returns has some characteristics of a non-normal generating process. In particular the results indicate that the distribution is fat-tailed relative to a normal distribution. However characteristics were also observed that were inconsistent with a stable non-normal generating process: a statistical model that is being used to represent the supposed random variations in observations. Officer (1972) noted that the distributions become thinner tailed for larger sums of daily stock returns but not to the extent that it approximates a normal distribution. Officer (1972) disagreed with some of the conclusions of Mandelbrot (1963): who examined the distribution of stock returns in the context of a non-normal stable Paretian distribution and Fama (1965): who noted that the distribution of monthly returns belonged to a non-normal member of the stable class of distributions, and found that the standard deviation appeared to be a well behaved measure of scale.

A distinguishing feature of symmetric non-normal stable class of distributions is peakedness and fat-tails when compared with the normal distribution. The parameter which measures the degree of peakedness and the fatness of the tails of stable distributions is the characteristic exponent, also known as the stability parameter, α. The range of characteristic exponents, for which a probability density function is known in closed form, is bounded by the normal distribution, for $\alpha = 2$, and the Cauchy distribution: that does not have a mean, variance or higher moments defined, for $\alpha = 1$. That is the range of the characteristic exponent is bounded by $1 \leq \alpha \leq 2$.

Officer (1972) examined the stationary process: a stochastic process whose unconditional joint probability distribution does not change when shifted in time; consequently parameters such as mean and variance also do not change over time, of the mean estimate of the characteristic exponent $\hat{\hat{\alpha}}$: an average of the n estimates $\hat{\alpha}$ of α, namely

$$\hat{\hat{\alpha}} = 1/n \sum_{j=1}^{n} \hat{\alpha}_j \, , \tag{4.26}$$

and therefore the constancy of α of the distribution of stock returns through time. These properties are central to determining whether the stock returns distribution can be approximated by a member of the stable class of distributions.

Initially Officer (1972) estimated the distribution of monthly returns of a random sample of $n = 39$ stocks listed continuously from January 1926 to June 1968: 509 observations. Officer (1972) reported that the distribution of monthly returns indicated a non-normal distribution with an estimated characteristic exponent $\hat{\alpha}$ of about 1.51.

Officer (1972) took the daily stock returns from a random 50 stocks taken from the sample of

136 stock examined that were listed over the entire sample period from June 2, 1962, to November 6, 1969. This period was split into eight sub-periods each with 217 observations. Officer (1972) found that the mean estimated characteristic exponent $\hat{\bar{\alpha}}$, varied between 1.61 and 1.68 for the sub-periods and each had a standard deviation of approximately 0.15. Officer (1972) concluded that the distribution of stock returns, defined by the estimated characteristic exponent $\hat{\alpha}$ values, had not changed substantially over the periods.

The central issue for Officer (1972) in describing the properties of the distribution of stock returns: stability and scale parameters, was determining whether the process generating the distribution of stock returns was a non-normal sable distribution. If the stock returns are distributed by a stable distribution whose parameters are constant over time, then the results should show smaller $\hat{\alpha}$'s for larger intervals. Alternatively if the process was one suggested by Press (1967): returns are distributed as if drawn from a normal distribution with changing scale parameter: standard deviation, then $\hat{\alpha}$'s should increase with the length of interval and should approach 2.0 for larger sums.

An additional check on the appropriateness of the supposed stability for small sums requires an examination of the parameters of the distribution. An important parameter is the scale parameter which measures the degree of dispersion of the distribution. If the scale parameter behaves in a predictable fashion then any measure of risk that is a function of the scale parameter is independent of the daily, monthly, yearly, *etc* time interval. An estimate of scale parameter is obtained from the sample fractile: the cut-off point where the distribution reaches a certain probability, and is independent of the stable distribution for $1.0 < \alpha < 2.0$. Officer (1972) found that the scale parameter is invariant for sums of stock returns up to five months as expected. What is more the standard deviation is well behaved. A further test on the behaviour of the standard deviation as a measure of scale was performed on the daily returns. Because daily tests are conducted over a shorter time span they are less likely than monthly tests to run into the problem of non-constancy of the parameters of the distribution. The results show a tendency for the $\hat{\alpha}$ to increase slightly for larger sums, where the reverse was expected. The fact that the $\hat{\alpha}$ of sums of daily returns do increase suggests a modified model with a finite second moment for the distributions. The behaviour of alternative measures of scale for the distributions of sums of daily stock returns was also tested. Once again the standard deviation appeared to be a good measure of scale and superior to other measures considered.

However these results do not mean that it is inappropriate to use a non-normal stable distribution to approximate the distribution of stock returns. The distributions examined have fat-tails so that the normal distribution is going to give a poor approximation of the distribution of returns. The well behaved standard deviation suggest the distributions will have some properties which non-normal stable distributions do not have. It may be that a class of fat-tailed distributions with finite second moments will give a better approximation of the distribution of stock returns.

Officer (1972) tested the stable Paretian and compound events models hypothesis for monthly and daily return data by computing the characteristic exponent α and observing its changes over the sample path.

For monthly returns Officer (1972) found that the behaviour of α was inconsistent for both models. The stable Paretian model predicts a smaller measured α over larger time intervals, and the compound events model predicts an increase in the measure of α to $\alpha = 2$, which is the characteristic of a normal distribution. Officer (1972) states that for monthly returns α appeared to be constant, evidence that not much is lost by assuming the distribution of monthly returns is stable.

For daily returns Officer (1972) found a slight increase in α suggesting that a modified model with a finite second moment for the return distribution might be appropriate, supporting the argument for the compound events model. Further support is provided by the fact the standard deviation of the daily returns seems well-behaved.

The findings on the daily returns are important from the portfolio manager's point of view who is likely to re-hedge more frequently than once a month. Consequently it is the distribution of daily returns that is important from an option pricing perspective.

4.2.2 The t-distribution of Returns

A class of fat-tailed distributions with finite second moments that give a better approximation to the distribution of stock returns is the Praetz (1972) scaled t-distribution. This distribution is the only known simple distribution to fit changes in share prices. It provides a better fit to the data than the Mandelbrot (1963) stable Paretian, the Press (1967) compound events process and the normal distribution.

Praetz (1972) modified a theory of the distribution of share price changes derived by Osborne (1959) by extending the Brownian motion principle of share price changes to account for the changing variance of the stock market. This produces a scaled t-distribution that is a good fit to series of share price indices.

If $S(t)$ represents the price of a share at time t, then the change in the logarithmic share price from time t to $t+\tau$ is given by

$$z = \ln S(t+\tau) - \ln S(t)\,. \tag{4.27}$$

Osborne (1959) notes that the prices can be interpreted as an aggregate of decisions in statistical equilibrium where the equilibrium distribution function of z is given by

$$f(z) = \frac{e^{(-z^2/2\sigma^2\tau)}}{\sqrt{2\pi\sigma^2\tau}}\,, \tag{4.28}$$

where σ^2 is the variance of z over unit time intervals. This distribution is the same as that of a particle in Brownian motion, thus the price $S(t)$ follows a geometric Brownian motion. The theory of Brownian motion used by Osborne (1959) implies the values of z over non-overlapping intervals of time constitute a random walk: the values of z are mutually independent and have a common probability function. Praetz (1972) was concerned with the common probability distribution function of the random walk rather than the independence of price changes.

Osborne (1959) implied that the distribution of share price changes should be normally distributed. However the evidence accumulated concludes that the distribution appears to be non-normal. Fama (1965), Madan and Seneta (1990), Mandelbrot (1963), Press (1967) and Kon (1984) conclude that the distribution of share price returns, relative to the normal distribution, is a symmetrical distribution with fat-tails, high peaked centre and hollow in between.

Praetz (1972) conducted a χ^2 goodness-of-fit test plus a test based on the third and fourth sample moment of skewness and kurtosis and concluded that the series of share price changes were not normally distributed. Praetz (1972) noted that the stock indices are well fitted by the scaled t-distribution at the one percent level of significance. However for individual stock prices Praetz (1972) noted that the situation is not as promising due to the discrete nature and the occurrence of a large number of zero price changes, rejecting the normal distribution proposed by Osborne (1959).

Osborne (1959) assumed that the variance of price changes over unit time interval, σ^2, is a constant. In practice this is not so. The financial markets have periods of activity followed by periods of inactivity ensuring the information that affects prices does not arrive uniformly but in bursts. In Brownian motion σ^2 is proportional to the degree of activity and evidenced by σ^2 varying as the degree of activity in the market varies.

Praetz (1972) suggested that the probability density function of the price variable z be conditioned on the value of σ^2. Taking a unity time interval $\tau = 1$ for simplicity and z having a non-zero mean μ, then the probability density function becomes

$$f(z|\sigma^2) = \frac{e^{(-(z-\mu)^2/2\sigma^2)}}{\sqrt{2\pi\sigma^2}} . \tag{4.29}$$

If $h(z)$ is the distribution of z that takes into account the random nature and distribution of σ^2, then $h(z)$ is given by

$$h(z) = \int_0^\infty f(z|\sigma^2)\, g(\sigma^2)\, d\sigma^2 \quad \text{with} \quad 0 \leq \sigma < \infty . \tag{4.30}$$

Praetz (1972) suggests the distribution of σ^2, the variance of price changes, is a random variable with distribution function $g(\sigma^2)$ given by

$$g(\sigma^2) = \frac{\bar{\sigma}^{2m}(m-1)^m \sigma^{-2(m+1)} e^{-(m-1)\bar{\sigma}^2/\sigma^2}}{\Gamma(m)} , \tag{4.31}$$

where the expected return $\bar{\sigma}^2 = E[\sigma^2]$ and variance of σ^2 is $\bar{\sigma}^4/(m-2)$ and m is the degrees of freedom. This is known as an inverted gamma (Γ) distribution. When $g(\sigma^2)$ in Equation (4.31) is substituted into Equation (4.30) then $h(z)$ is obtained by integrating as

$$h(z) = [1 + (z-\mu)^2/\bar{\sigma}^2(2m-2)]^{-m-1/2}\Gamma(m)[(2m-m)\pi]^{1/2}\bar{\sigma} . \tag{4.32}$$

This is a t-distribution of $2m = n$ degrees of freedom, where n is the sample size from a normally distributed population with expected mean value μ and variance σ^2, except for a scale factor $[n/(n-$

$2)]^{1/2}$. Thus the distribution of $(z-\mu)/\bar{\sigma}$ is a scaled t-distribution. Therefore the distribution of the share price $S(\tau)$ at time τ can be obtained from $z = \ln S(\tau) - \ln S(0)$. Thus giving

$$f[S(\tau)] = \frac{\{1 + [\ln S(\tau) - \ln S(0) - \mu\tau]^2/\bar{\sigma}^2\tau(2m-2)\}^{-m-1/2}\Gamma(m+1/2)}{\bar{\sigma}\tau^{1/2}S(\tau)\Gamma(m)\sqrt{(2m-2)\pi}}, \tag{4.33}$$

where z has a mean of $\mu\tau$ and variance of $\sigma^2\tau$ over a time interval τ and $S(0)$ denotes the price at time 0.

The distribution function $g(\sigma^2)$ of the variance has a mean $\bar{\sigma}^2$, variance $\bar{\sigma}^4/(m-2)$ and a mode at $\bar{\sigma}^2(m-1)/(m+1)$. It is 0 at $\sigma^2 = 0$, rises to a peak and has a long tail to the right. This represents the distribution of the variance of a share price and reflects the changing market expectations of investors.

For the share price indices the t-distribution describes the changes in logarithmic prices. Thus enabling explicit probability statements concerning the changes in price. These probabilities are larger for large changes than those of a normal distribution. However for individual prices the situation is not as promising. The distributions used by Mandelbrot (1963) to represent price changes are between the Cauchy and a normal distribution. The Praetz (1972) scaled t-distribution also lies between the same two distributions. Hence the t-distribution can be considered as an alternative to the stable Paretian model.

Praetz (1972) proposed a stochastic framework for the distribution of the volatility parameter σ^2. Rejecting the normal distribution advanced by Osborne (1959): share price changes follow on from the Brownian motion rationale, and concluding that the distribution is in fact non-normal, comprising a symmetrical distribution with fat-tails, a higher peaked centre and hollow in-between compared to a normal distribution.

4.2.3 The Empirical Evidence on Return Distributions

There have been a number of investigations into the statistical qualities of stock price returns: Officer (1972), Madan and Seneta (1990), Mandelbrot (1963), Fama (1965), Press (1967), Praetz (1972) and Kon (1984) that include analytical models for stock price changes that are relevant for option pricing. Accepting the conclusions drawn by Officer (1972) and Praetz (1972) that a class of fat-tailed distributions with finite second moments will give a better approximation of the distribution of stock returns, a comprehensive description and explanation of the stock price returns that adhere to this theory are presented here for the:

i. Variance Gamma Model.

ii. Stable Paretian Distribution.

iii. Compound Events Model.

iv. Discrete Mixture of Normal Distributions.

The models considered proposed an alternative to the normal distribution for the stock returns. Acknowledging Officer (1972) and Praetz (1972) the models will be reviewed in light of the elements that describe the foreign exchange (FX) market delineated in Chapter 3: order flow, bid-ask spread and price triangulation, to determine which model can incorporate the system of attributes to explain the FX market.

The variance gamma model falls into the category that respecifies the stochastic process for the stock price volatility, $\sigma(S, t)$, whereas the stable Paretian, compound event and discrete mixture of normal distributions models specify an alternative stochastic process for the stock price return.

The latter category is of a particular interest. This category can be applied to the Black and Scholes (1973) method to arrive at the differential equation which may be solved using the boundary conditions defined by the option intrinsic value.

The Variance Gamma Model

Madan and Seneta (1990) proposed the variance gamma model for stock returns as a practical alternative to the role of Brownian motion. The variance gamma model is an extension of Brownian motion, obtained by evaluating a normal process at a random time defined by a gamma process. Thus the variance gamma model replaces the time in the Brownian motion with a gamma process. The variance gamma process is proposed as a model for the uncertainty underlying security prices. The model states that the unit period distribution is normal conditional on a variance that is distributed as a gamma variate: a class of random variable associated with a gamma function. The advantages of the variance gamma model include long tails, continuous-time specification and finite moments of all orders. The process is a pure jump process where small jumps occur very frequently and large jumps occur only occasionally and approximated by a compound Poisson process: where the jumps arrive randomly according to a Poisson process and the random size of the jumps is specified by a probability distribution, with high jump frequencies and low jump magnitudes.

The variance gamma model proposed by Madan and Seneta (1990) satisfies the following empirical properties:

i. Long tails relative to the normal distribution for daily returns, with returns over longer periods approaching normality: Fama (1965).

ii. Finite moments for lower powers of returns.

iii. Consistent continuous-time stochastic process with independent stationary increments belonging to the same family of distributions irrespective of the length of time.

The variance gamma model is well placed in meeting the criterion that satisfies the findings of Officer (1972) when compared to other models. Conversely Brownian motion does not comply with the first property, the Mandelbrot (1963) stable Paretian model fails on the second and third and the Praetz (1972) t-distribution fails on the third point. Although the Poisson distribution mixture of normal distributions of the Press (1967) compound events model possesses all the properties described above, the Madan and Seneta (1990) variance gamma model has a further advantage of being a pure jump process of a large number of small jumps. Madan and Seneta (1990) state that the variance gamma model is a limit of a particular sequence of compound events models in which the arrival rate of jumps approaches infinity, while the magnitudes of the jumps are concentrated near the origin. The variance gamma model respects the intuition underlying the continuous sample path of Brownian motion as a model.

When considering the empirical relevance of the variance gamma model for stock market returns Madan and Seneta (1990) compared the variance gamma model with the Mandelbrot (1963) stable Paretian distribution, the Press (1967) compound event model and the normal distribution using a χ^2 goodness-of-fit statistic on seven class-intervals for unit sample variance data on nineteen stocks quoted on the Sydney Stock Exchange. Madan and Seneta (1990) found that the variance gamma model outperformed the other models in attaining the minimum χ^2 statistic twelve out of nineteen times, followed by the compound events model with seven times and five times for the stable Paretian distribution and two for the normal distribution.

The variance gamma is obtained from the normal by the variance-mean mixture: a probability distribution of a random variable that is derived from a collection of other random variables. Hence it comprises a continuous probability distribution of a random variable g that changes over unit time and follows a gamma process with mean rate 1, variance rate ν and mixing probability density $g(\nu)$. The probability density function (pdf) is a weighted average of a family of pdf's where the weight is the density function of the random variable. Let $R(t)$ be the return over a unit time period, stated formally as

$$R(t) = \frac{S(t+1)}{S(t)}\,, \tag{4.34}$$

taking logarithms and rearranging:

$$\ln R(t) = \ln S(t+1) - \ln S(t)\,, \tag{4.35}$$

where $S(t)$ is the stock price at time t and $\ln R(t)$ is normally distributed with mean μ and random variance $\sigma^2 V$, where μ and σ^2 are known constants and V follows a gamma process with mean rate of 1 and variance rate of ν. The distribution of V is taken to be a two-parameter gamma distribution: a continuous probability distribution defined in terms of its gamma function given by

$$g(\nu) = \frac{c^\gamma\,\nu^{\gamma-1}\,e^{-c\nu}}{\Gamma(\gamma)}\,, \tag{4.36}$$

where c is the scale parameter: the larger the scale parameter the more spread out the distribution and γ is the shape parameter: affects the shape of the distribution estimated using higher moments such as skewness and kurtosis, $g(\nu)$ the probability density and Γ is the gamma function extension of the factorial function to real numbers. More generally, for any positive real number γ, $\Gamma(\gamma)$ is defined as

$$\Gamma(\gamma) = \int_0^\infty x^{\gamma-1} e^{-x}\,dx \qquad \text{for} \quad x > 0\,. \tag{4.37}$$

If random variable $X = \ln(R) - \mu$ is normally distributed with mean zero and variance one, then the probability density function of a normal variance-mean mixture of X, $f(x)$, with mixing probability density $g(\nu)$ is given by

$$f(x) = \int_0^\infty \frac{e^{(-x^2/(2\sigma^2\nu))}}{\sigma\sqrt{2\pi\nu}}\, g(\nu)\, d\nu\,. \tag{4.38}$$

Equation (4.38) has no closed-form expression. However the characteristic function for the probability distribution of X, $\phi_x(u)$, has a closed-form expression obtained by conditioning on V

$$\phi_x(u) = \left[1 + (\sigma^2 v/m)(u^2/2)\right]^{-m^2/v}, \tag{4.39}$$

where $m = \gamma/c$ is the mean of the gamma density $g(\nu)$ and $u = \gamma/c^2$ is its variance. From the form of the characteristic function only $(\sigma^2 v/m)$ and $-m^2/v$ are identified. Given that σ^2 serves as the scale parameter for V, the mean of V can be taken to be unity: $m = 1$ or $\gamma = c$.

Madan and Seneta (1990) show that the variable V can be viewed as a random time change and setting $m = 1$ or $\gamma = c$ is consistent with supposing that the expected random time change is unity for the unit period return. The characteristic function of the unit period return distribution therefore becomes

$$\phi_x(u) = \left[1 + (\sigma^2 v u^2/2)\right]^{-1/v}\,. \tag{4.40}$$

The higher moments of the variance gamma distribution are obtained by conditioning on V. The variance gamma has finite moments of all orders and in particular the second and fourth moments which are given by

$$EX^2 = \sigma^2 \quad \text{and} \quad EX^4 = 3\sigma^4(1+v)\,. \tag{4.41}$$

The kurtosis is therefore $3(1+v)$. Under normality the kurtosis is 3. The proportional excess of the kurtosis over 3 is v, and may be regarded as a measure of the degree of long tailedness.

Madan and Seneta (1990) considered the effects of varying v on the density function of the unit period variance gamma distribution. Madan and Seneta (1990) noted that as the degree of long tailedness, v, increases the distribution of the density function, increasing the probability near the origin: becoming more peaked, as well as to increase the tail probabilities: fatter-tailed, at the expense of the intermediate range.

If g is the change over time T in a random variable that follows a gamma process with mean rate 1 and variance rate of ν then the variance gamma model can be characterised by letting g define the rate at which information arrives during time T. If g is large, a great deal of information arrives and the distribution has a large mean and variance, conversely if g is small, little information arrives and the mean and variance are small.

Madan and Seneta (1990) introduced a multivariate extension of the variance gamma model by letting X denote a vector of random variables distributed conditional on the non-negative random variable V as a multivariate normal with mean vector zero and variance-covariance matrix ΣV. By conditioning on V, the joint characteristic function of X, $\phi_x(u)$ where u is a vector, is given as

$$\phi_x(u) = \left(1 + vu^T \Sigma u/2\right)^{-1/v}, \tag{4.42}$$

which generalises the univariate characteristic function, see Equation (4.40). Given that ϕ_x is a function of u via the quadratic form $u^T \Sigma u$, the joint density is elliptical and the conditional expectation function of X_i is linear.

A shortcoming of the multivariate model is that the measure of kurtosis, v, is the same for all distributions and hence have the same kurtosis.

The variance gamma meets the non-normal, higher peaked and fatter-tailed criterion with finite moments of all orders, in particular the second, set out by Officer (1972) and Praetz (1972). However the univariate gamma model does not lend itself to applying the FX attributes: order flow, bid-ask spread and triangulation, as a system determining FX price return and is disregard on that basis for modelling the FX market.

Conversely the multivariate model may be considered by applying the FX attributes as a vector of random variables notwithstanding the limitation of this application due to the constant measure of kurtosis for all the distributions.

The Stable Paretian Distribution

Mandelbrot (1963) presented a new model of price behaviour in speculative markets. The feature of this model is starting from the Bachelier (1900) process as applied to $\ln S(t)$ instead of $S(t)$. The model replaces the Gaussian distributions throughout by another family of probability laws, referred to as stable Paretian: Lévy (1925). The Gaussian is a limiting case of the stable Paretian and as such is a generalisation of the Bachelier (1900) process. However, even the logarithmic form of the Bachelier (1900) random walk model inadequately describes the price changes on the basis that the tails of the distribution appear to be too long to be accounted for by a normal distribution. Empirical distributions of price changes are too peaked to be relative samples from Gaussian populations: Officer (1972) and Praetz (1972). Mandelbrot (1963) noted that the histogram of price changes was indeed unimodal. However the distribution of outliers fitted to the variance of the price changes are much lower and flatter than the distribution of the data itself. Mandelbrot (1963) remarked that the tails of the distribution of price changes were in fact so extraordinarily long that the sample second moments typically varied in an erratic fashion. The stable Paretian distribution with infinite second moments replaces the Gaussian normal with well behaved variance in the Mandelbrot (1963) model.

Mandelbrot (1963) posed a non-Gaussian stable process with a finite mean but an infinite variance as a possible explanation for the erratic behaviour of the variances of successive price changes. Fama (1965) supports the hypothesis of non-zero mean, long-tailed, peaked, non-Gaussian distributions for logarithmic price changes. However, Cootner (1962) and Godfrey, Granger, and Morgenstern

(1964) took an opposing view. On examining securities from a time and frequency point of view they concluded that there was no evidence in any series that the process which generated them behaved as if it possessed an infinite variance. Mandelbrot (1963) attempted to address this by proposing a model for the behaviour of security price changes in the mathematics of stable laws.

Since the stable Paretian probability laws are relatively unknown, Mandelbrot (1963) discusses the important mathematical properties of these laws in Appendix C on page 157.

Let $S(t)$ be the price of a stock at the end of time period t. Successive differences of the form $S(t+T) - S(t)$ are independent Gaussian or normally distributed random variables with zero mean and variance proportional to the differencing interval T. Noting that the empirical distribution of the prices changes were too peaked relative to the normal with long tails and irregular second moments, Mandelbrot (1963) presented the Lévy stable distribution as an alternative potential model for returns. The stable distributions are a class of distributions where a linear combination of two independent random variables has the same distribution as the individual variables and satisfies the following relationship

$$a_1 U + a_2 U \stackrel{\Delta}{=} aU \quad \text{where} \quad a = a_1 + a_2\,, \tag{4.43}$$

where U is a random variable, a_1 and a_2 represent scale factors closely related to the variance in the Gaussian case, and a is a function of a_1 and a_2. The normal distribution satisfies the above relation, as does the Cauchy distribution: a distribution that does not have a mean, variance or moment generating function defined. The normal distribution is the only stable distribution with a finite variance. The Cauchy distribution, on the other hand, has an infinite second moment but a finite expectation. All stable distributions with a finite expectation can be thought of as lying on a continuum between the Cauchy distribution and the Gaussian.

The standard Gaussian variable, with zero mean and unit variance, is a solution to Equation (4.43). The Gaussian variable expresses an unchanging consistency under addition, hence it is referred to as being stable: a linear combination of two independent random variables has the same distribution, up to location and scale parameters. The normal distribution satisfies this relationship as the only stable distribution with a finite variance. When the variance is allowed to be infinite the scale factors are not defined by any moment.

Mandelbrot (1963) proposed the alternative Paretian distribution, a power-law: change in one quantity resulting in a proportional change in another quantity, skewed probability distribution with heavy, slow decaying tails that contain large amounts of the data. Further, Fama (1965) noted the empirical distributions of price changes are leptokurtic: higher peaks and fatter-tails than a normal distribution with the same mean and variance, rejecting the hypothesis that price changes are normally distributed. Mandelbrot (1963) and Fama (1965) agreed that the daily change in the logarithmic prices of stocks follow a stable Paretian distribution with characteristic exponent or tail index power value $\alpha \simeq 2$.

Fama (1965) noted that successive price changes are independent, supporting the application of the Central Limit Theorem: where the mean of a sample of data having any distribution converges upon a normal distribution as the sample size tends to infinity, to the price change or return distribution. However, the distribution of price differences, or logarithmic price differences, might not be identical for successive price changes. The distribution of price changes may have a volatility that is a function of the stock price as in geometric Brownian motion, *see* Equation (B.15) on page 156. Officer (1972) took exception to the conclusions of Mandelbrot (1963) and Fama (1965). Although Officer (1972) found that the distribution of stock returns were fat-tailed relative to a normal distribution, he also observed characteristics that were inconsistent with non-normal processes. Officer (1972) highlighted that the daily stock returns became thinner tailed for larger sums but not sufficiently to display a normal distribution. Accepting the Fama (1965) assertion of the independence of successive price changes the focus will be on the form of the distribution of price changes.

Mandelbrot (1963) proposed the stock distribution to model logarithmic price changes for a unit time interval that preserves the convenient feature of the Gaussian model: that the various increments depend only upon T where

$$L(t,T) = \ln S(t+T) - \ln S(t)\,, \tag{4.44}$$

is a Gaussian random variable for every value of T: price increments over days, weeks, months, and years would have the same distribution. The only thing that changes with T is the standard deviation of $L(t,T)$ and the increment $L(t,1) = \ln S(t+1) - \ln S(t)$ is a random variable with infinite population moments beyond the first. This implies the density function $f(u)$ for random variable U is such that

$$\int\limits_{-\infty}^{\infty} f(u)\, u^2\, du \quad \text{diverges but} \quad \int\limits_{-\infty}^{\infty} f(u)\, u\, du \quad \text{converges}\,. \tag{4.45}$$

It is natural to assume the density function $f(u)$ is well behaved: functions which have derivatives of all orders at all points and which, together with their derivatives, fall off at least as rapidly as u^{-n} as $u \longrightarrow \infty$, no matter how large n is. Therefore for large u, as $u \longrightarrow \infty$, $f(u)\, u^3$ tends to ∞ and $f(u)\, u^2$ tends to zero. If the second moment of the logarithmic price changes diverge and the first moment is well behaved then density function $f(u)$ must decrease faster than u^{-2} but slower than u^{-3}. The simplest expressions of this type are those with an asymptotically Paretian behaviour.

Such stable processes do not have finite variance. This has been posed as a possible explanation for the erratic behaviour of the variances of successive price changes observed empirically. Alternative models in which prices have finite second moments might explain price fluctuation variances at least as well, and at the same time might not require that all previous portfolio selection theory based upon finite variance distributions and quadratic loss functions to be abandoned. Such concern has been expressed by Cootner (1964), who also feels the sample evidence (based on cotton prices) is not at all compelling.

Officer (1972) and Praetz (1972) noted that a class of fat-tailed distributions with finite second moments will give a better approximation of the distribution of stock returns. The Mandelbrot (1963) stable Paretian model proposes a skewed probability distribution with heavy, slow decaying, fat-tails in accordance with Officer (1972) and Praetz (1972). However the model deviates from the requirement of finite higher moments. Mandelbrot (1963) suggested an infinite second moment as an explanation of the erratic behaviour of the variances of successive price changes. Based on this fundamental divergence from the criterion set out above, the Mandelbrot (1963) stable Paretian distribution can be excluded as a potential model, incorporating the systemic attributes, for the FX market.

The Compound Events Model

Press (1967) advocated a statistical model for the distribution of security price changes: the compound events model, where the logarithmic price changes are assumed to be independent random walks. The model proposed is distinguished in that the logarithmic price changes are not assumed to follow a stable distribution which could be normal. Instead the logarithmic price changes are assumed to follow a Poisson distribution mixture of normal distributions. Such a distribution is skewed and leptokurtic: more peaked at its mean than a comparable normal distribution, and has greater probability mass in its tails than the distribution of a comparable normal variate, aligning with the price change distribution outlined by Mandelbrot (1963) and Fama (1965).

Press (1967) noted that there is no need to conclude that the variance is infinite, because of non-zero higher-order cumulants: quantities that provide an alternative to the moments of the distribution, observations will be found further from the mean and that the modal ordinate will be higher, more peaked, than would be expected on the basis of normal theory.

The compound events model can be interpreted conceptually by asserting that the price of a security can be accounted for by aggregating a random number of price changes of random size: represented as a Poisson distribution, which take place during the time interval observed, then superimposing Brownian motion. For the Press (1967) model very large variances in the change in logarithmic price levels will be obtained if many changes take place during the interval of interest. The distribution associated with this model is leptokurtic: higher peaks and fatter-tails than a normal distribution with the same mean and variance.

The basic, single security, Press (1967) compound events model involves a Poisson process with Gaussian jump sizes stated as

$$Z(t) \equiv \ln S(t) = \ln S(0) + \sum_{k=1}^{N(t)} Y_k + X(t) \quad t = 1, 2, \dots, \tag{4.46}$$

where:

- $S(t)$ is the price of a given security at time t and $Z(t)$ denotes the natural logarithm of the price of a given security at time (t), assumed to be stationary independent increments composed of a compound Poisson process: the jumps arrive randomly according to a Poisson process and the random size of the jumps is specified by a probability distribution, augmented by a Wiener process and $S(0)$ is assumed known.
- $Y_1, Y_2, \dots, Y_k$ is a sequence of mutually independent random variables drawn from a normal distribution with mean θ_{ce} and variance σ_2^2, obeying $\sim N(\theta_{ce}, \sigma_2^2)$.
- $N(t)$ is a Poisson process with parameter λt, which represents the number of random price changing events occurring in time t where $\{N(t), t \geq 0\}$ is independent of Y_k and $\{X(t), t \geq 0\}$ is a Wiener process independent of $N(t)$ and of $Y_1, Y_2, \dots, Y_k$.

- $X(t) \sim N(0, \sigma_1^2 t)$ has stationary and independent increments drawn from a normal distribution with mean 0 and variance $\sigma_1^2 t$.

By differencing Equation (4.46) gives

$$\Delta Z(t) \equiv Z(t) - Z(0) = \sum_{k=N(t-1)+1}^{N(t)} Y_k + \epsilon(t) \quad t = 1, 2, \dots, \tag{4.47}$$

where $\epsilon(t) = X(t) - X(0)$ is the stationary independent normal process, $\sim N(0, \sigma_1^2)$. This model resembles random walk models in that it is a discrete parameter, continuous state space, Markov process and $X(t)$ is a Brownian motion. The uniqueness of the Press (1967) model is that the compound Poisson process: $\sum_1^{N(t)} Y_k$, produces a non-Gaussian $Z(t)$ process, and for $\lambda = \infty$, $Z(t)$ must follow some stable law.

Summarising the properties of the distribution characteristics of $Z(t)$ and $\Delta Z(t)$, see below, Press (1967) noted that they seem to be in agreement with the empirically determined properties of security price changes and stated in the lemmas contained in Appendix D: *Lemmas - Compound Events Model* on page 159.

Let $\phi^*(u)$ denote the characteristic function denoting the probability distribution of the one-step price change $\Delta Z(t)$ as defined in Equation (4.47). For very small λ, $\phi^*(u)$ is normally distributed. For large λ, the higher order terms in u produce a substantial departure from normality. The mean and variance for $Z(t)$ and $\Delta Z(t)$ are

$$\begin{aligned}
E\left[Z(t)\right] &= Z(0) + \theta_{ce}\lambda t\,, \\
Var\left[Z(t)\right] &= [\sigma_1^2 + \lambda(\theta_{ce}^2 + \sigma_2^2)]t \quad \text{and} \\
E\left[\Delta Z(t)\right] &= \theta_{ce}\lambda\,, \\
Var\left[\Delta Z(t)\right] &= \sigma_1^2 + \lambda(\theta_{ce}^2 + \sigma_2^2)\,,
\end{aligned} \tag{4.48}$$

where $Z(t)$ is a process of stationary and independent increments and $Var\left[\Delta Z(t)\right]$ is a linearly increasing function of λ. Therefore the variance of the relative logarithmic prices tends to grow as the mean number of price-changing events increases.

The distribution of $\Delta Z(t)$ is leptokurtic and when the mean θ_{ce} is small, the probability in the extreme tails of the distribution of $\Delta Z(t)$ exceeds that of a comparably normally distributed random variable. From *Lemma (3)* on page 159, the first four cumulants of the distribution of $\Delta Z(t)$ are given as K_1, K_2, K_3 and K_4, where K_4 is positive, which is what is needed for leptokurtosis.

The distribution of $\Delta Z(t)$ is more peaked in the vicinity of its mean than a comparable normal random variable. The density function in the vicinity of the mean of the distribution of $\Delta Z(t)$ is greater than the density of a standard normal variable.

The distribution of $\Delta Z(t)$ is symmetric about its mean if its mean is zero; otherwise, the distribution is asymmetric. The skewness, γ_1, of the distribution of $\Delta Z(t)$ is obtained where the skewness has the same sign as that of the mean θ_{ce}. Moreover, $\gamma_1 = 0$ if and only if the mean $\theta_{ce} = 0$, that is, if and only if $E[\Delta Z(t)] = 0$.

When the mean $|\theta_{ce}|$ is small, the probability in the extreme tails of the distribution of $\Delta Z(t)$ exceeds that of a comparable normally distributed variable. The implications are that the density of $\Delta Z(t)$ exceeds that of a standardised normal random variable as long as the densities are compared for all points beyond some point chosen sufficiently far out in the tails of the distributions.

Press (1967) extended the basic security model to incorporate a multivariate statistical model to study the joint behaviour of a group of securities. Examinations of the joint behaviour of many securities found that securities tend to move in groups: King (1966). Press (1967) incorporated these findings in defining the multivariate model given as

$$Z(t) = C + X(t) + H(t) + \sum_{k=1}^{M(t)} W_k \,, \tag{4.49}$$

where C, X, H and W_k are $p-$variate vectors with:

$$\begin{aligned} Z(t) &= [Z_1(t), \ldots, Z_p(t)] \,, \\ X(t) &= [X_1(t), \ldots, X_p(t)] \,, \\ W_k &= [W_{k1}, \ldots, W_{kp}] \,, \\ H(t) &= \left[\sum_{k=1}^{N_1(t)} Y_{k1}, \ldots, \sum_{k=1}^{N_p(t)} Y_{kp} \right] , \\ C &= [Z_1(0), \ldots, Z_p(0)] \equiv [C_1, \ldots, C_p] \,, \end{aligned} \tag{4.50}$$

where:

- $Z_j(t)$ is a vector process of stationary and independent increments and denotes the natural logarithm of the price of the jth security at time t.
- $X_j(t)$ denotes the mutually independent Wiener process associated with the jth security, $j = 1, \ldots, p$ and distributed as $\sim N(0, t\sigma^2)$.
- C is the vector of initial prices.
- $M(t)$ represents the number of price changing events that affect all securities from $(0, t]$ and $M(t)$ is assumed to have a Poisson distribution with parameter $\bar{\lambda}t$.

- W_k is assumed to be mutually independent and independent of $M(t)$, and to follow the multivariate normal distribution $\sim N(\mu, \Sigma)$ for every k.
- $H(t)$ are assumed to be independent and contain mutually independent Poisson processes with parameters $\lambda_j t, j = 1, \ldots, p$ which represents the number of price changing event in $(0, t]$ that affect each security.
- Y_{kj} are distributed as $\sim N(\theta_{cej}, \sigma_j^2)$ and assumed to be mutually independent for all k and all $j = 1, \ldots, p$.
- $X(t), H(t)$ and $\sum_1^{M(t)} W_k$ are assumed to be mutually independent.

The logarithmic price of each security comprises of an initial price, a linear combination of price changes correlated with the market as a whole and some random price fluctuations representative of Brownian motion.

The single security model is obtained from the multivariate model by setting $p = 1$ and $\bar{\lambda} = 0$.

Press (1967) tested his compound events model graphically on the price change distributions of several stocks as follows:

i. Computed and graphed the cumulative distribution function of actual stock price changes.

ii. Estimated the parameters of the model (θ_{ce}, λ), and of the skewness γ_1, the kurtosis γ_2, the first four moments (m_k), and the first four cumulants (K_k).

iii. Plotted the estimated theoretical cumulative distribution function on the same graph as the actual cumulative distribution function.

The compound events model of a Poisson distribution mixture of normal distributions satisfies the requirements of Officer (1972) and Praetz (1972) to identify a class of distribution that give a better approximation of stock returns. The Press (1967) model is more peaked with a greater mass in its tails than a comparable normal distribution and with a finite variance. The Press (1967) model is concerned with aggregating a random number of logarithmic price changes of a random size occurring during an observed time interval rather than the information flow driving the price changes. The univariate model can be disregarded because it does not incorporate the system of attributes to explain the FX market. The multivariate model describes the joint behaviour of logarithmic price changes of a group of securities and can be considered to model the FX triangle prices operating as a group. However the application to the FX market is limited. The model does not allow for the systemic information flow but rather accounts for changes occurring during an observed time interval in describing the FX market.

The compound events models possesses the right properties, but must be extended and calibrated with greater accuracy if it is to provide a workable model for FX option pricing. In contrast to the stable Paretian distribution, the compound events model possess the theoretical advantage of having a finite second moment, allowing the use of standard statistical theory. The models offer some scope to model an aspect of the FX market but are not sufficient to cover all the attributes required.

The Discrete Mixture of Normal Distributions

Kon (1984) proposed a discrete mixture of normal distributions to explain the positive skewness and kurtosis: a measure of the fat-tails for a probability distribution of a real-valued random variable, in the distribution of daily rates of return for common stock and stock indices. Kon (1984) noted that stationary tests on the parameter estimates, where the mean, variance and autocorrelation are all constant over time, revealed significant differences in the mean and variance estimates that can explain the observed skewness and kurtosis, respectively, of the daily return distribution.

Kon (1984) compared the discrete mixture of normal distribution models with the Praetz (1972) t-distribution model and concluded that the discrete mixture of normal distribution model had *"substantially more descriptive validity"*.

Kon (1984) was concerned with the description of stock returns. The form of the distribution of the stock returns is a crucial element for mean-variance portfolio theory, capital asset pricing models and contingent claims. The most convenient assumption for empirical models is that the distribution of stock returns is multivariate normal with parameters that are stationary over time. Given that the normal distribution is stable under addition, any portfolio of stocks will also be normally distributed. The assumptions of normality; crucial to models of financial theory and the BSM equation, and stationarity are fundamental to the econometric techniques of empirical research.

Evidence from Blattberg and Gonedes (1974) indicated that the distribution of monthly returns conforms to normality and hence empirical work proceeded to use monthly data. Research to isolate information sets or utilise the advantage of a large sample for statistical reliability in asset pricing models requires the use of daily data.

Fama (1965) tested the normality of the daily returns of the Dow Jones Industrial stocks and found more kurtosis: fatter-tails, than predicted from i.i.d. normal variates. Fama (1965) concluded that the distribution of price changes conforms to a stable distribution with characteristic exponent $\alpha < 2$.

The empirical evidence on the distribution of daily stock returns rejects the stationary normal distribution model: Fama (1965). Normality is crucial to models of financial theory whereas stationarity is a convenient sampling assumption.

Changes in the financial decision variables results in the adjustments to the expected returns and standard deviation parameters of the returns distribution. An alternative explanation for the fat-tails observed in stock returns involves the model specification. Consider a generating process of stock returns as a mixture of normals where the variance is a random variable.

Changing the specification of the normal distribution model, A. J. Boness, Chen, and Jatusipitak (1974) found that with weekly return data the parameters of the price change process shifted before and after a capital structure change. The series departed from normality more frequently than in the periods of pre- or post- capital structure changes. Further, A. A. Christie (1982) found that the standard deviation of stock returns is an increasing function of the leverage effect. Seasonal signals also lead the parameter shifts. Beaver (1968) noted that seasonal announcements result in rate of return observations with higher variance during the disclosure periods.

A. A. Christie (1982) formulated a discrete mixture of two normal distributions to model returns.

The higher variance distribution represented the information event while the other distribution is attributed to the non-information random variables. Ball and Torous (1983) derived a model comprising a mixture of two normal distributions for daily returns. The model resulted from a Bernoulli jump process: one scenario for failure and the other for success, to describe information arrivals where the Bernoulli distribution: discrete distribution with only two discrete scenarios, is used to set the occurrence ($x = 1$) or lack of occurrence ($x = 0$) to denote a jump in a stochastic variable. However, the generating process of stock returns is further complicated by exogenous macro information:

i. Firm-specific or market-wide.

ii. Institutional trading restrictions: price clustering, price discreteness and days of the week the trade occurs.

The information generating process is more nuanced than the two model scenarios considered. The process is drawn from a mixture of normal distributions. The actual number of normal distributions is an empirical issue and may vary across stocks or even currencies.

Kon (1984) considered the validity of the discrete mixture of normal distributions process as a statistical model for stock returns. Kon (1984) estimated the parameters: where differences in the mean estimates explain the skewness and differences in the variance estimates explain the kurtosis, of the respective models for mixtures of $N = 1, 2, 3, 4$ and 5 normal distributions. Taking a sample of 30 Dow Jones Industrial stocks Kon (1984) found that the likelihood-ratio test of the models specification indicated that 7 stocks can be described by a mixture of four normal distributions; three models described 11 stocks and two normal distributions for the remaining 12 stocks.

Kon (1984) suggested that the distribution of stock returns was normal with parameter shifts among a finite set of values. These parameter moves are due to *time-ordered shifts* associated with:

i. Capital structure changes.

ii. Acquisitions.

iii. Stock splits.

iv. Exogenous events.

And *cyclical-shifts* between sets of parameters due to:

v. The days of the week of the trades.

vi. Seasonal announcements.

vii. Earnings and dividends.

Kon (1984) assessed the potential impact of both types of shifts on the distribution by taking $4,639$ daily return observations from an $18\frac{1}{2}$-year time-series from July 2, 1962, to December 31, 1980, on the 30 stocks in the Dow Jones Industrial Average, the Standard and Poor's Composite (S&P), the Center for Research in Security Prices (CRSP) valued weighted (VW) and the CRSP equal-weighted (EW) indices partitioned by:

i. Year to account for time-ordered shifts.

ii. Days of the week to account for cyclical-ordered shifts.

iii. Year and days of the week to account for both effects.

To identify the parameters of the normal distribution Kon (1984) assumes each return observation is drawn from one of N sets of parameter values. These parameters accommodate both the cyclical and time-ordered structural shifts. The generalised discrete mixture of normal distribution model views each return observation on a stock, r_t, as being generated by one of the following N distinct equations

$$\begin{aligned} r_t &= \mu_1 + U_{1t} \quad \text{if } \ t \in I_1 \, , \\ r_t &= \mu_2 + U_{2t} \quad \text{if } \ t \in I_2 \, , \\ &\vdots \\ r_t &= \mu_N + U_{Nt} \quad \text{if } \ t \in I_N \, , \end{aligned} \tag{4.51}$$

where $I_i \; i = 1, 2, \ldots, N$ are the homogeneous information sets with T_i observations in each set, thus $\sum_{i=1}^{N} T_i = T$. The random variables U_{it} are independent and identically normally distributed with a mean of zero and variance of σ_i^2, $0 < \sigma_i^2 < \infty$, $i = 1, 2, \ldots, N$.

Defining $\lambda_{dmi} = T_i/T$ as the proportion of observations associated with information set I_i. Then, for a given N, the parameter vector $\theta_{dm} = \{\mu_1, \mu_2, \ldots, \mu_N, \, \sigma_1^2, \sigma_2^2, \ldots, \sigma_N^2, \, \lambda_{dm1}, \lambda_{dm2}, \ldots, \lambda_{dmN-1}\}$ can be estimated by maximising the likelihood function

$$l(\theta_{dm}|r) = \prod_{t=1}^{T} \left[\sum_{i=1}^{N} \lambda_{dmi} \, p(r_t|\gamma_i') \right] , \tag{4.52}$$

where $r = (r_1, r_2, \ldots, r_T)', \gamma_i' = (\mu_i, \sigma_i^2)$, and $p(r_t|\gamma_i)$ is a normal probability density function with mean μ_i and variance σ_i^2.

Given N, the maximum likelihood procedure estimates $3N - 1$ parameters defining the model specification.

Kon (1984) found that the entire sample period for all three indices exhibited significant skewness and kurtosis (fat-tails) at the 1% level of significance. However partitioning the data into annual sub-periods reduced the frequency of rejecting the stationary normal null hypothesis. Partitioning by day of the week still rejected the null hypothesis, but with less significance than for the entire sample period. Conversely partitioning the entire sample period by year *and* by day of the week further reduced the frequency of rejecting the stationary normal null hypothesis.

Kon (1984) concluded that the test of normality on the partition of the data by year *and* by day of the week was an incentive for developing the discrete mixture of normal distributions hypothesis. The mixture process accommodates both cyclical and time-ordered shifts in the two parameters of the normal distribution. Therefore a model intended to represent the mixture process must accommodate cyclical shifts in parameter values due to the days of the week or seasonal announcements and the structural time-ordered shifts associated with capital changes, stock splits or exogenous market events.

The Kon (1984) formulation of a discrete mixture of normal distributions to model returns satisfies the criterion outlined by Officer (1972) and Praetz (1972): a class of fat-tailed distributions with a finite variance will give a better approximation of stock returns. Further each return is drawn from a set of parameter values containing homogeneous information sets with numerous observations. This can be adapted and applied to the FX market attributes: order flow, bid-ask spread and triangulation, making up a system of information sets for each return. Thus applying the Kon (1984) discrete mixture of normal distributions model for stock returns to the foreign exchange market may offer an explanation of the market behaviour.

4.2.4 Conclusions

This chapter reviewed the models that offer an alternative to the normal distribution for the stock price returns. The models have been critiqued in light of the elements that describe the foreign exchange (FX) spot price: order flow, bid-ask spread and triangulation, to determine which model incorporates these attributes to explain the FX market.

This chapter considered the Kon (1984) discrete mixture of normal distributions and the Press (1967) compound events model as possible alternatives in specifying an improved stochastic process for the price changes. The discrete mixture of normal distributions model seemed the most promising. The distributions of the information events align themselves with the order flow, bid-ask spread and triangulation from the FX market. That is the model can incorporate the systemic elements of the FX spot market. Further, the Kon (1984) discrete mixture of normal distributions can be adapted to the Black and Scholes (1972) methodology in deriving an alternative FX option pricing model.

Of the remaining models, the Press (1967) compound event model has proven to be a reliable alternative to the Kon (1984) option but must be calibrated at a greater accuracy if it is to provide a workable solution. The Madan and Seneta (1990) variance gamma model specifies a stochastic process for the stock price volatility and is empirically consistent with stock returns where long tails for daily returns are present. The Mandelbrot (1963) stable Paretian distribution converges to a Gaussian behaviour over large time scales under the model's claim to represent price changes. However the model's infinite variance is at odds with Officer (1972) and Praetz (1972) and can be eliminated as a potential model of the FX market.

When adapting a model of stock price returns to explain the foreign exchange market, the criterion the model must comply with include the following characteristics:

i. Fat-tails and peaked relative to the normal distribution of the same mean and variance.

ii. Slow convergence to Gaussian behaviour of logarithmic returns.

iii. Non-constant mean and variance over time.

An eligible model incorporating these requirements can then be evaluated in light of the systemic attributes of the FX market: order flow, bid-ask spread and triangulation. The question to ask is whether the model can be applied to a system and not simply a linear solution?

The Kon (1984) model meets the systemic requirements of the FX spot market and is suitable for further investigation as a potential representation of a normally distributed FX market model.

Chapter 5

Stochastic FX Price Processes

The critical factor in the Black and Scholes (1973) and R. C. Merton (1973) (BSM) analysis was the precise description of the stochastic process governing the price behaviour of the underlying asset. Relaxing the price process assumption that stock prices follow a geometric Brownian motion, the stochastic process for foreign exchange, based on the stochastic models for stock price returns, will be respecified. The foreign exchange price process will then be applied to the BSM methodology to arrive at an appropriate differential equation.

Under the price process assumption of Officer (1972) and Praetz (1972) that a class of fat-tailed distributions with a finite second moment will give a better approximation of stock returns, the stochastic stock model applied to foreign exchange is conditioned on the transformation incorporating the systemic attributes of the foreign exchange (FX) market: order flow, bid-ask spread and triangulation.

The Kon (1984) discrete mixture of normal distributions model satisfied these requirements and will be adapted for the FX market to determine its appropriateness.

The remainder of this chapter is organised as follows. Section 5.1 discusses the parameter definitions and model specification of the discrete mixture of normal distributions. Section 5.2 discusses the data set and empirical methodology. Section 5.3 presents the results and 5.4 the conclusions.

5.1 Respecifying the Distribution of Foreign Exchange

5.1.1 The Discrete Mixture of Normal Distributions

This chapter examines the model proposed by Kon (1984) to explain the observed fat-tails and significant positive skewness in the distribution of daily rates of return: the discrete mixture of normal distributions. Stationarity tests on the parameter estimates of this model revealed significant differences in the mean and variance estimates that can explain the observed skewness and kurtosis: fat-tails, respectively.

5.1.2 Parameter Definition and Model Specification

The generalised discrete mixture of normal distribution model views each return observation on a stock, r_t, as being generated by one of N distinct equations; *see* Equation (4.51) on page 65.

Given the T observations on the stock return variable, r_t, there exists a permutation of the rows of $r = (r_1, r_2, \ldots, r_T)'$ which will be partitioned according to Equation (4.51). In the presence of a multinomial prior: the number of distinct permutations of a multiset of n elements before some evidence is taken into account, the equation associated with the information set I_i is selected for generating observations with probability λ_{dmi}, $i = 1, 2, \ldots, N$. Thus each observation is viewed as being drawn from a mixture distribution: the probability distribution of a random variable derived from a collection of other random variables. The conditional probability density function (p.d.f.): the value of any point in the sample space can be interpreted as providing a relative likelihood that the value of a random variable would equal that point, of r_t given the parameter vector θ_{dm} is given by

$$f_t(r_t|\theta_{dm}) = \sum_{i=1}^{N} \lambda_{dmi} p(r_t|\gamma_i') \,, \tag{5.1}$$

where:

- $\gamma_i' = (\mu_i, \sigma_i^2), i = 1, 2, \ldots, N$, and
- $\theta_{dm} = \{\mu_1, \mu_2, \ldots, \mu_N,\ \sigma_1^2, \sigma_2^2, \ldots, \sigma_N^2,\ \lambda_{dm1}, \lambda_{dm2}, \ldots, \lambda_{dmN-1}\}$ is the parameter vector to be estimated subject to the sample space Ω
- $\Omega = \{\theta_{dm}; -\infty < \mu_i < \infty,\ 0 < \sigma_i^2 < \infty,\ i = 1, 2, \ldots, N,\ 0 < \lambda_{dmi} < 1,\ i = 1, 2, \ldots, N-1\}$ and
- $p(r_t|\gamma_i')$ is a normal probability density function with mean μ_i and variance σ_i^2, $i = 1, 2, \ldots, N$ and
- $\sum_{i=1}^{N} \lambda_{dmi} = 1$; $\sigma_1^2 < \sigma_2^2 < \cdots < \sigma_N^2$; and $T_i \geq 2$, $i = 1, 2, \ldots, N$.

The parameter vector θ_{dm} is estimated by choosing values that maximise the likelihood function

$$l(\theta_{dm}|r_t) = \prod_{t=1}^{T} \left[\sum_{i=1}^{N} \lambda_{dmi} p(r_t|\gamma_i') \right] . \tag{5.2}$$

The generality of the likelihood function in Equation (5.2) can be seen when comparing it to the true likelihood function given by

$$l(\theta_{dm}|r_t) = \prod_{t=1}^{T_1} p(r_t|\gamma_1^*) \cdot \prod_{t=1}^{T_2} p(r_t|\gamma_2^*) \cdots \prod_{t=1}^{T_N} p(r_t|\gamma_N^*) \,. \tag{5.3}$$

Under the assumption that the parameters and the partition of the T observations into T_i, $i = 1, 2, \ldots, N$ is known *a priori*, then no proportionalities are necessary: that they are multiplicatively connected to a constant. However, information about the partition is not known *a priori* and hence the true likelihood function, Equation (5.3), cannot be applied directly. To achieve the required result the right hand side of the general likelihood function, Equations (5.2), can be expanded to obtain

$$l(\theta_{dm}|r_t) = \sum_{T_1=0}^{T}\sum_{T_2=0}^{T}\cdots\sum_{T_N=0}^{T}\frac{T!}{T_1!\ldots T_N!}\lambda_{dm}{}_1^{T_1}\lambda_{dm}{}_2^{T_2},\cdots,\lambda_{dm}{}_N^{T_N}h_{(T_1,T_2,\ldots,T_n)}(\gamma_1',\gamma_2',\ldots,\gamma_N')\,, \quad (5.4)$$

where $(T_1 + T_2 + \cdots + T_N = T)$ and

$$h_{(T_1,T_2,\ldots,T_n)}(\gamma_1',\gamma_2,\ldots,\gamma_N) = \prod_{t=0}^{T_1}p(r_t)|\gamma_1')\cdot\prod_{t=0}^{T_2}p(r_t|\gamma_2')\cdots\prod_{t=0}^{T_N}p(r_t|\gamma_N')\,. \quad (5.5)$$

The summation in Equation (5.4) is over all feasible values of $T_1, T_2, \ldots, T_N$. Given each set of feasible values the summation is therefore over all possible ways of partitioning the r vector into the vectors $r_{(t_i)}, i = 1, 2, \ldots, N$ with $T_i, i = 1, 2, \ldots, N$ observations in each vector. Therefore, any scenario that generates data from N normal probability distributions *in any order* is included as a subset of the general specification of Equations (5.2).

For a given N and T, the maximum likelihood estimator θ_{dmT} of θ_{dm} is defined by the supremum: the least element that is greater than or equal to all elements of the set, of the logarithmic likelihood given by

$$L_T(\theta_{dm}) = \sum_{t=1}^{T}\ln f_t(r_t|\theta_{dm})\,. \quad (5.6)$$

Then the maximum likelihood estimator θ_{dmT} is the solution to the likelihood equations

$$\frac{\partial L_T(\theta_{dm})}{\partial\theta_{dm}} = \sum_{t=1}^{T}\frac{1}{f_t(r_t|\theta_{dm})}\cdot\frac{\partial f_t(r_t|\theta_{dm})}{\partial\theta_{dm}} = 0\,, \quad (5.7)$$

such that the matrix of second partials evaluated at the solution point is negative definite in which all eigenvalues are negative. The elements $\partial L_T(\theta_{dm})/\partial\theta_{dm}$ in Equation (5.7) are nonlinear in θ_{dm} so that no closed form solution for θ_{dmT} exists. However Equation (5.7) represents an implicit solution for the θ_{dmT}. Therefore employing the iterative gradient method identifies the local maxima in finding the solutions satisfying the maximum maximorum.

Hypothesis tests on the elements of the estimated parameter vector, θ_{dmT}, can be constructed with the information in the sample covariance matrix. For maximum likelihood estimates, the sample covariance matrix is the negative inverse of the matrix of the second partial derivatives of the logarithmic likelihood function with respect to the parameter vector evaluated at $\theta_{dm} = \theta_{dmT}$.

Maximisation by Quadratic Hill-Climbing Iterative Gradient Method

Goldfeld, Quandt, and Trotter (1966) proposed a new gradient method for maximising general functions. The new algorithm maximises a quadratic approximation to the function on a suitably chosen spherical region. The method requires no assumptions about the concavity of the function and automatically modifies the step size.

A variety of problems reduce to maximising or minimising functions of several variables. This type of problem is dependent on the computation of the maximum likelihood estimates of the coefficients which requires maximising the likelihood function or its logarithm.

Maximisation techniques take the form of iterative processes; given a point corresponding to a set of values for the independent variable, a new point at which the function is larger is computed. Repetition of the process leads to a point which converges to the location of the maximum. Convergence requires the assumption that the function is concave in the region of the computed points. Convergence can be guaranteed provided the initial value is sufficiently close to the maximum. In the absence of *a priori* knowledge of the function there is no way to ensure the starting point will satisfy such conditions. The Goldfeld et al. (1966) proposed method is designed to work for functions which are not concave everywhere and for starting points which may not be near the maximum.

Consider a function $H(x_1, \ldots, x_n)$, denoted by $H(x)$, of n variables to be maximised. Let x denote the column vector of variables $(x_1, \ldots, x_n)$, F_x the first partial derivatives evaluated at x and the symmetric matrix S_x express the second partial derivative evaluated at x. To maximise $H(x)$ requires choosing a starting point $x^o = (x_1^o, \ldots, x_n^o)$ and iterating according to

$$x^{p+1} = x^p + h^p D^p \,, \tag{5.8}$$

where h^p is a positive constant and D^p is an n-dimensional direction vector. In gradient methods the choice of D^p is given by

$$D^p = B^{-1} F_{x^p} \,, \tag{5.9}$$

where B is a positive definite weighting matrix: a symmetric matrix where every eigenvalue is positive and F_{x^p} is the gradient of F evaluated at x^p.

A simple choice for B is given by setting $B = I$ where I is the identity matrix: an $n \times n$ square matrix with ones on the main diagonal and zeros elsewhere. This choice of B derives the method of steepest ascent. The rational behind this value of B, and hence of D^p, is that the gradient points in the direction of the maximum increase of the best local linear approximations to $H(x)$.

Assume that $H(x)$ recognises a second-order Taylor series expansion around a point $a = (a_1, \ldots, a_n)$ such that

$$H(x) \approx H(a) + (x-a)' F_a + \frac{1}{2}(x-a)' S_a (x-a) \,, \tag{5.10}$$

where the subscripts indicate the point of evaluation. Equation (5.10) corresponds to a first-order expansion for the first partials obtained by differentiating with respect to x, that is

$$F_x \approx F_a + S_a(x-a)\,. \tag{5.11}$$

The method of steepest ascent implies

$$x^{p+1} = x^p + h^p F_{x^p}\,, \tag{5.12}$$

substituting Equation (5.12) into (5.10), replacing a by x^p and omitting the subscripts, gives

$$H(x^{p+1}) - H(x^p) = h^p F'\,F + \frac{1}{2}(h^p)^2(F'\,SF)\,. \tag{5.13}$$

Choosing h^p so as to maximise Equation (5.13) yields the "optimum" gradient method. Treating Equation (5.13) as a functions of h^p, say $G(h^p)$, gives

$$\begin{aligned} \frac{dG}{dh^p} &= F'\,F + h^p(F'\,SF) = 0 \\ \text{or} \qquad & \\ h^p &= -(F'\,SF)^{-1}F'\,F\,. \end{aligned} \tag{5.14}$$

In order that this value of h^p yields a maximum then

$$\frac{d^2G}{d(h^p)^2} = F'\,SF < 0\,, \tag{5.15}$$

which is necessarily so if S is negative definite: in which all eigenvalues are negative. If x^p is not sufficiently close to the maximum to assure that S_{x^p} is negative definite, this procedure may fail.

In practice the "optimum" gradient choice of h^p has not worked well and alternatives have been used.

The difficulties associated with the steepest ascent method leads to the second most common version of the gradient method: Newton's method. This method is obtained by maximising Equation (5.10) with respect to x. Setting $x^p = a$ and Equation (5.11) to zero, gives the iterative scheme

$$x^{p+1} = x^p - S_{x^p}^{-1}\,F_{x^p}\,. \tag{5.16}$$

This is a gradient method with $h^p \equiv 1$ and $B = -S_{x^p}^{-1}$. Equation (5.16) might require taking a step so large that the quadratic approximation based on the behaviour of the function X^p: S_{x^p} and F_{x^p}, has no validity at x^{p+1}. In addition S_{x^p} may be negative definite and hence does not have a maximum. While attempts have been made to solve the problem of non-negative definiteness of S, no satisfactory solution has been proposed.

Goldfeld et al. (1966) proposed a new method which uses the same quadratic approximations but includes a parameter which limits the size of the step taken.

Consider the quadratic function $Q(x)$ where the matrix S is constant and the expansions in Equations (5.10) and (5.11) are exact. If S is non-singular: one that has a matrix inverse, then from Equation (5.11)

$$c = a - S^{-1}\, F_a\,, \tag{5.17}$$

where c is a unique point where $F_x = 0$. If S is negative definite, Q has a unique global maximum at c. If S is non-singular, Q is not bounded above.

Let $\|x\|$ denote the length of the vector x, defined as $(x'\,x)^{\frac{1}{2}}$. Thus $\|x - y\|$ is the distance between x and y.

Goldfeld et al. (1966) determined the properties of the iterative process stated in the following lemmas.

Lemma 1: Let α be any number such that $S - \alpha I$ is negative definite, and define

$$b_\alpha = a - (S - \alpha I)^{-1}\, F_a \tag{5.18}$$

and

$$r_\alpha = \|b_\alpha - a\|\,. \tag{5.19}$$

Then $Q(b_\alpha) \geq Q(x)$ for all x such that $\|x - a\| = r_\alpha$.

Lemma 2: If $F_a \neq 0$ then r_α defined by Equations (5.18) and (5.19) is a strictly decreasing function of α on the interval (λ_1, ∞) where λ_1 is the maximum eigenvalue of S.

Let α, b_α and r_α be as in Equations (5.18) and (5.19). Let B_α be the region consisting of all x such that $\|x - a\| \leq r_\alpha$, and suppose $F_a \neq 0$. Then the maximum value of $Q(x)$ on B_α is attained at b_α if $\alpha \geq 0$, and is attained at b_0 if $\alpha < 0$.

If S is negative definite, Q has an absolute maximum at b_α. Since $\lambda_1 < 0$, both 0 and α are in the interval (λ_1, ∞) and by $\|b_0 - a\| \leq \|b_\alpha - a\|$ if and only if $\alpha < 0$. Thus if $\alpha < 0$, b_0 is in the interior of B_α and the maximum of Q on B_α occurs at b_0. However if $\alpha \geq 0$, b_0 is not in the interior of region B_α and there is no local maximum of Q in the interior. Hence when S is not negative definite the maximum on B_α must occur at b_α.

Lemma 3: If $F_a = 0$, then the maximum value of Q on the region B_r, consisting of all x with $\|x - a\| \leq r$ occurs at $a \pm r u_1$ if λ_1 is positive and at a otherwise, where u_1 is a unit eigenvector associated with λ_1.

The Goldfeld et al. (1966) iterative procedure for finding the maximum given point x^p at which S_{x^p} and F_{x^p} are evaluated defines the next point, x^{p+1}, as the maximum of the quadratic approximation, *see* Equation (5.10), on a spherical region centred at x^p. Ideally the region should be as large as possible provided that in the region the quadratic approximation is a satisfactory guide to the actual behaviour of the function. In approximating this ideal, two distinct cases arise:

i. F_{x^p} significantly different from 0.

In this event choose a number

$$\alpha = \lambda_1 + R\|F_{x^p}\|\,, \tag{5.20}$$

where λ_1 is the largest eigenvalues of S_{x^p}, and R is a positive parameter to be described below. Now take

$$\begin{aligned} x^{p+1} &= x^p - (S_{x^p} - \alpha I)^{-1}\, F_{x^p} \\ \text{or} \qquad & \\ x^{p+1} &= x^p - S_{x^p}^{-1}\, F_{x^p}\,, \end{aligned} \tag{5.21}$$

according to whether α is positive or not. x^{p+1} is the maximum of the quadratic approximation to the function on a region B_α of radius $\|(S_{x^p} - \alpha I)^{-1}\, F_{x^p}\|$ with centre at x^p. The larger the value of α the larger the value of R and the smaller the size of the region B_α. Hence the radius of B_α is

$$\|(S_{x^p} - \alpha I)^{-1}\, F\| \leq (\|F_{x^p}\|R)^{-1}\|F_{x^p}\| = R^{-1}\,, \tag{5.22}$$

it is reasonable to expect the two quantities, $\|(S_{x^p} - \alpha I)^{-1}\, F_{x^p}\|$ and R^{-1}, will in general be of the same order of magnitude. In practice an initial value of R which appears reasonable is given to the algorithm and then R is automatically modified at each iteration. Given the value of α one computes a new iteration and accepts the step if the actual change in the function is positive. If the function deteriorates, R is increased so as to take smaller steps which are repeated until an improvement is obtained.

ii. F_{x^p} is so near 0 that the length of the step taken is within a pre-set tolerance of 0.

If S_{x^p} is negative definite, the process is terminated and x^p is accepted as the location of the maximum. If S_{x^p} is not negative definite, this is either a saddle-point or at the bottom of the maximum then *Lemma 3* applies. A step is taken along the eigenvector corresponding to λ_1 and the algorithm recycles in the usual manner.

One final feature was the introduction of a scalar h^p into Equation (5.21) as

$$x^{p+1} = x^p - h^p(S_{x^p} - \alpha I)^{-1}F_{x^p}\,. \tag{5.23}$$

At each step the computation is first performed with $h^p = 1$. If this gives an improvement in $H(x)$, h^p is multiplied by a constant and the function is examined at the new point obtained. This process is repeated until the function declines in which case the last step is accepted.

5.1.3 Model Specification Tests

For each stock, five model specifications were considered for $N = 1, 2, 3, 4$ and 5. For each N the logarithmic likelihood for the normal mixture was maximised by the Goldfeld et al. (1966) modified quadratic hill-climbing algorithm, *see* Section 5.1.2. A comparison of the stationary normal distribution with a mixture of two normals, $N = 2$, can be made with the pairwise test between different specifications using the generalised likelihood-ratio

$$LR = -2\ln\Lambda_{ij} = -2\ln\left[\frac{l(\theta_{dmi}|r)}{l(\theta_{dmj}|r)}\right] \quad \text{where} \quad i < j\,, \tag{5.24}$$

where the quantity inside the square brackets is the likelihood-ratio. As all likelihoods are positive, and as the constrained maximum cannot exceed the unconstrained maximum the likelihood-ratio is bounded between zero and one. The smaller the likelihood-ratio the larger the χ^2 will be. The null hypothesis is rejected if χ^2 is larger than a χ^2-percentile with 3 degrees of freedom. The likelihood-ratio test can also be expressed as the difference between the logarithmic likelihoods as

$$LR = -2\ln\Lambda_{ij} = -2\left[\ln l(\theta_{dmi}|r) - \ln l(\theta_{dmj}|r)\right] \quad \text{where} \quad i < j\,. \tag{5.25}$$

Multiplying by -2 ensures mathematically that LR converges asymptotically to being χ^2-distributed if the null hypothesis is true.

For example, the likelihood-ratio test for the stationary normal model against the alternative hypothesis of a mixture of two normal distributions is $\Lambda_{12} = l(\theta_1|r)/l(\theta_2|r)$. The significance tests for discriminating between these hypotheses can be constructed from noting that the asymptotic distribution of $-2\ln\Lambda_{ij}$ is a χ^2 with degrees of freedom equal to the difference in the number of parameters between the two models: 2 parameters for the general normal distribution: mean and variance, and 5 parameters for the mixture of two normal distributions: two means, two variances and the mixing parameter, giving the 3 degrees of freedom. The stationary normal distribution null hypothesis is rejected in favour of the mixture of two normals at the 5% probability level when the test statistic exceeds 7.815.

5.2 Data Set and Methodology

The Refinitiv Eikon trading system, formerly Thomson-Reuters, provided the data for this study. The data set covers the 10 year period from February 19th, 2010, to February 19th, 2020. The data consists of 2,609 observations for each of the top four currencies by percentage share of average daily turnover as per the April 2019 BIS survey; *see* Table 3.3 on page 31. The related currency pairs consists of EUR/USD, EUR/GBP, GBP/USD, USD/JPY, EUR/JPY and GBP/JPY, *see* Table 3.2 on page

31. The exchange rate price history comprises the best bid, mid and ask quotes and time-stamped to the daily closing price. No information as to the transaction size or trading parties is given.

Analysis of Skewness and Kurtosis

Given that the skewness and excess kurtosis of the normal distribution are zero, values for these two parameters should be close to zero for data to follow a normal distribution.

The skewness test statistic utilised by Kon (1984) incorporates the third central moment divided by the three-halves power of the second central moment. Furthermore the kurtosis coefficient is the fourth central moment divided by the square of the second central moment. This implies that the sample data can tell us something about the skewness and kurtosis of the population data. The process outlined by Kon (1984) will not be followed here, rather the standard error of skewness and kurtosis as a measure of the separation from zero as recommended by Cramer (2002) will be applied and outlined in Appendix F: *Statistical Analysis of Skewness and Kurtosis* on page 189.

To test the null hypothesis that the distribution is normal the D'Agostino-Pearson omnibus test will be applied. This test incorporates the test statistics for both skewness and kurtosis to come up with a single p-value. The test statistic follows a χ^2 distribution with 2 degrees of freedom: 5.9915 at the 5% level of significance, and given by

$$DP = Z_{Sk}^2 + Z_{Ku}^2 \,. \tag{5.26}$$

Note that the D'Agostino-Pearson test has a tendency to err on the side of rejecting normality, particularly with small sample sizes of below twenty.

5.3 Results

The statistical test for normality for the entire sample period is reported in Table 5.1. For brevity the statistics for the sub-partitioned data are not disclosed here but are available in Appendix E: *Test for Departure from Normality* on page 161.

TABLE 5.1: Currency Pairs - Entire Sample Period Test for Departure from Normality

Exch. Rate	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
EUR/GBP	0.0000	0.0051	0.6005	12.5274	8.1741	85.2909	7431.4769	0.0000	2608
EUR/USD	−0.0001	0.0055	−0.0286	−0.5970	1.9594	20.4449	418.3517	0.0000	2608
GBP/USD	−0.0001	0.0055	−1.3535	−28.2349	21.9361	228.8878	53 186.8389	0.0000	2608
USD/JPY	0.0001	0.0056	−0.1463	−3.0522	5.1745	53.9919	2924.4390	0.0000	2608
EUR/JPY	0.0000	0.0067	−0.3755	−7.8322	6.1295	63.9576	4151.9140	0.0000	2608
GBP/JPY	0.0000	0.0074	−1.8470	−38.5297	32.2177	336.1693	114 494.3049	0.0000	2608

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

Kon (1984) noted that stocks exhibited significant skewness and kurtosis at the 1% probability level for the entire sample period, foreign exchange seems to be equally unequivocal.

The statistics presented in Table 5.1 assume the returns for each currency pair are independent and identically distributed.

TABLE 5.2: Currency Pairs - Results: Test for Departure from Normality

Exch. Rate	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	Normality
EUR/GBP	Moderately Skewed	Positive	Leptokurtic	Leptokurtic	*Reject*
EUR/USD	Approx. Symmetric	Inconclusive	Leptokurtic	Leptokurtic	*Reject*
GBP/USD	Highly Skewed	Negative	Leptokurtic	Leptokurtic	*Reject*
USD/JPY	Approx. Symmetric	Negative	Leptokurtic	Leptokurtic	*Reject*
EUR/JPY	Approx. Symmetric	Negative	Leptokurtic	Leptokurtic	*Reject*
GBP/JPY	Highly Skewed	Negative	Leptokurtic	Leptokurtic	*Reject*

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis.

From Tables 5.1 and 5.2 the observed sample skewness S_k suggests that GBP/USD and GBP/JPY are highly negatively skewed, EUR/GBP is moderately positively skewed and EUR/USD, USD/JPY and EUR/JPY are approximately symmetric. The skewness test statistic Z_{Sk} exhibits a probable negative population skewness for all currency pairs with the exception of EUR/GBP which displays a positive population skewness and EUR/USD which is inconclusive: possibly positively or negatively skewed or not skewed, at the 5% level of significance. The excess kurtosis Ku for the entire sample period for all currency pairs is leptokurtic: a higher peak at the mean and fatter-tails than a normal distribution with the same mean and variance. The kurtosis test statistic Z_{Ku} suggests the population kurtosis is also leptokurtic at the 5% level of significance, indicating fatter-tails than the normal distribution. The D'Agostino-Pearson test rejects the null hypothesis assumption of normality for all the currency pairs.

Therefore the normality test concludes that for the sample period the population return for foreign exchange exhibited significant skewness and kurtosis at the 5% probability level. The distribution is negatively skewed with a higher, sharper central peak and longer, fatter-tails when compared to a normal distribution with the same mean and variance. The results in Table 5.1 and 5.2 clearly reject the stationary normal distribution hypothesis for each currency pair for the entire sample. The discrete mixture of normal distributions may be able to explain these results.

Kon (1984) noted that partitioning the data reduced the frequency of rejecting the stationary normality null hypothesis for stock. Partitioning lead to a reduction in the kurtosis test statistic which is improved still further the greater the partitioning applied.

In order to determine if the skewness and kurtosis for foreign exchange gives a better indication of the population normality of the foreign exchange returns, further partitioning was applied to the sample data, *see* Appendix E on page 161. The function used to determine the standard errors of skewness, Z_{Sk}, and kurtosis, Z_{Ku}, assumes the data is drawn from a normal distribution. This suggests that if the statistics are more than 1.96 standard errors from the hypothesized values at the 5% level of significance the value can be rejected. This is only relevant when the standard error estimate and the sampling distribution are approximately normal. Evaluating the skewness and kurtosis for each partitioned sub-period, the corresponding test statistic was applied to affect the probable population skewness and kurtosis. Although there is no minimum size for the test, a note of caution as the sample sizes get smaller through sub-partitioning, the test becomes more sensitive to the assumption that the samples are drawn from populations with a normal distribution. The data was partitioned into the following sub-periods:

i. Annual.

ii. Day of the week.

iii. Both year *and* day of the week.

For brevity the findings are displayed in Appendix E on page 161.

The findings show that partitioning the data by year the measure of skewness S_k identifies the sample data as approximately symmetric. Conversely the population skewness test statistic Z_{Sk} is inconclusive: the population being either symmetric or skewed in either direction at the 5% level of significance. The excess kurtosis Ku for the yearly partitioned data is leptokurtic and the population kurtosis test statistic Z_{Ku} suggests the population is also leptokurtic at the 5% level of significance. The D'Agostino-Pearson test rejects the null hypothesis assumption of normality but less frequently than the test for the entire sample period in Table 5.2.

Partitioning the data by day of the week the measure of skewness S_k identifies the sample data as approximately symmetric. For the majority of the population the skewness test statistic Z_{Sk} is inconclusive: the population being either symmetric or skewed in either direction with a minority negatively skewed at the 5% level of significance. The excess kurtosis Ku is leptokurtic and the test statistic Z_{Ku} suggests the population is also leptokurtic at the 5% level of significance. The D'Agostino-Pearson test rejects the null hypothesis assumption of normality at the same frequency as the entire sample shown in Table 5.2.

Partitioning the data by year *and* day of the week the measure of skewness S_k identifies the sample data as approximately symmetric. The population skewness test statistic Z_{Sk} is predominately inconclusive: the population being either symmetric or skewed in either direction with only a minority negatively skewed at the 5% level of significance. The excess kurtosis Ku for each partitioned sub-period was leptokurtic for the majority and platykurtic for the minority of returns at the 5% level of significance. The D'Agostino-Pearson test does not reject the null hypothesis assumption of normality.

When evaluating the normality of the distribution Kon (1984) suggested partitioning the data into annual sub-periods reduced the frequency of rejecting the stationary normal null hypothesis. Applying the D'Agostino-Pearson normality test to the sub-periods concurred with these findings. The partition by year still rejected the null hypothesis but with less significance than for the entire period whereas partitioning by day of the week did not have an effect. Applying both partitions of year *and* by day of the week lead to a reduction in the skewness Z_{Sk} and kurtosis Z_{Ku} test statistics and the D'Agostino-Pearson test does not reject the null hypothesis assumption of normality.

Kon (1984) noted that the observed skewness may be explained by shifts in the mean parameter in the time-series and the observed kurtosis: fat-tails, are consistent with shifts in the variance parameter. The tests of normality on the data partitioned by year *and* day of the week provide a strong motivation for pursuing the Kon (1984) discrete mixture of normal distributions to model the foreign exchange returns.

For each currency pair six potential model specifications: $N = 1, \ldots, 6$ will be considered. Given N, the logarithmic likelihood of the normal was maximised by the modified quadratic hill-climbing algorithm set out in Section 5.1.2 on page 71. A comparison of the stationary normal distribution with a mixture of N normal distributions was made utilising the likelihood-ratio test given by Equation (5.25) on page 75. This statistic has an asymptotically χ^2 distribution with 3 degrees of freedom: 7.815 at the 5% level of significance, equal to the difference in the number of parameters between the models: the sum of the N means, N variances and $N - 1$ mixing parameters for each model specification. In order to reject the stationary normal distribution null hypothesis in favour of the mixture of N normal distributions at the 5% probability level, the test statistic must exceed 7.815.

To examine whether the parameters of the models are stable across the various sub-samples of the data a multiple breakpoints test was used: a test for parameter instability and structural change to the models $N = 1, \ldots, 6$. For each N model the quadratic hill-climbing iterative gradient method was applied to determine the logarithmic likelihood for that model, *see* Table 5.3.

TABLE 5.3: Currency Pairs - Model Specification Logarithmic Likelihood

Model N	Param.	EUR/GBP	EUR/USD	GBP/USD	USD/JPY	EUR/JPY	GBP/JPY
$N = 1$	2	10 064.850	9868.453	9867.460	9808.375	9340.327	9098.510
$N = 2$	5	10 069.160	9870.985	9869.917	9813.950	9344.514	9102.603
$N = 3$	8	10 072.140	9876.949	9875.474	9818.645	9349.971	9108.539
$N = 4$	11	10 073.820	9879.212	9877.088	9822.064	9351.168	9110.928
$N = 5$	14	10 075.080	9881.264	9879.238	9824.187	9351.816	9111.977
$N = 6$	17	10 075.870	9882.191	9879.823	9825.557	9351.540	9112.047

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For each currency pair the statistic $-2\ln\Lambda_{ij}$ is used to test the null hypothesis of $N = i$ against the mixture of $N = j$ normal distributions, *see* Table 5.4.

TABLE 5.4: Currency Pairs - Model Specification Logarithmic Likelihood-Ratio Test

Λ_{ij}	EUR/GBP	EUR/USD	GBP/USD	USD/JPY	EUR/JPY	GBP/JPY
Λ_{12}	8.620	5.064	4.914	11.150	8.374	8.186
Λ_{23}	5.960	11.928	11.114	9.390	10.914	11.872
Λ_{34}	3.360	4.526	3.228	6.838	2.394	4.778
Λ_{45}	2.520	4.104	4.300	4.246	1.296	2.098
Λ_{56}	1.580	1.854	1.170	2.740	−0.552	0.140

χ^2 three degrees of freedom 7.815 at 5% level of significance.
χ^2 three degrees of freedom 11.345 at 1% level of significance.

The likelihood-ratio test, $-2[\ln\Lambda_1 - \ln\Lambda_2]$, for the stationary normal null hypothesis, $N = 1$, against the alternate hypothesis of a mixture of two normal distributions, $N = 2$, has in the main greater values than the 7.815 at 3 degrees of freedom required to reject $N = 1$ in favour of $N = 2$ for EUR/GBP, USD/JPY, EUR/JPY and GBP/JPY with only EUR/USD and GBP/USD not rejecting the null hypothesis.

Further, of the four currency pairs that had rejected the null of $N = 1$ in favour of $N = 2$ at the 5% level of significance, three currency pairs: USD/JPY, EUR/JPY and GBP/JPY rejected the two normal distributions, $N = 2$, in favour of three normal distributions, $N = 3$ at the 5% level of significance.

The mixture of $N = 4, 5$ and 6 was attempted on the entire sample and no satisfactory optimum was obtained for any of the currency pairs.

To verify whether the parameter shifts support the discrete mixture of normal distributions as a model for foreign exchange returns the individual difference test for each parameter was undertaken.

TABLE 5.5: Currency Pairs - Parameter Differencing

Model N	EUR/GBP		EUR/USD		GBP/USD	
	mean	var.	mean	var.	mean	var.
$N = 2$	0.0000	0.0000	−0.0001	0.0000	−0.0001	0.0000
$N = 3$	0.0001	0.0001	−0.0002	0.0001	−0.0002	0.0001
$N = 4$	0.0005	0.0001	−0.0003	0.0001	−0.0004	0.0001
Model N	USD/JPY		EUR/JPY		GBP/JPY	
	mean	var.	mean	var.	mean	var.
$N = 2$	0.0001	0.0000	−0.0001	0.0001	−0.0001	0.0001
$N = 3$	0.0004	0.0001	−0.0001	0.0001	−0.0004	0.0001
$N = 4$	0.0003	0.0001	−0.0002	0.0001	0.0000	0.0002

All models reject the unit root null hypothesis at the 1% level of significance.

Note that transformations such as logarithms can help to stabilise the variance of a time series.

Differencing a time series can help to stabilise the mean by removing changes in the level of a time series and therefore eliminating trend and seasonality effects. The former was applied to the data here. Kon (1984) noted that changes in the financial decision variables resulted in adjustments to the mean and standard deviation parameters of the distribution of the return. The parameters shift along a finite set of values to explain the skewness due to differences in the mean and the kurtosis due to shifts in the variance parameter.

For each currency pair and model specification in Table 5.5 the unit root null hypothesis was rejected at the 1% level of significance and the parameter estimates were differenced. The difference test of the stationarity hypothesis for the mixture of two, three and four normal distributions models for each currency pair are shown in Table 5.5. For EUR/GBP, which was identified with two normal distributions, the mean and variance stationarity is rejected for $N = 3$ and 4. For EUR/USD and GBP/USD, both identified with a single normal distribution, the mean stationarity is rejected for $N = 2, 3$ and 4 and the variance stationarity for $N = 2$ and 4, for both models. For the models identified by three normal distributions: (USD/JPY, EUR/JPY and GBP/JPY), USD/JPY rejects the mean stationarity for $N = 2, 3$ and 4 and the variance stationarity for $N = 3$ and 4, EUR/JPY rejects the mean and variance stationarity for $N = 2, 3$ and 4 and GBP/JPY rejects the mean stationarity for $N = 2$ and 3 and the variance stationarity for $N = 2, 3$ and 4. This agrees with Table 5.2 to explain the skewness, which varied from moderately positively skewed to approximately symmetric to highly negatively skewed and the kurtosis which was consistently leptokurtic throughout for the entire sample period.

The parameter stationarity test and model specification support the discrete mixture of normal distributions as a model of foreign exchange returns. Concluding that at the 5% level of significance the sample may be described by three normal distributions for USD/JPY, EUR/JPY and GBP/JPY and two normal distributions for EUR/GBP with EUR/USD and GBP/USD described by one normal distribution.

Note that the currency pairs involving the Japanese Yen are described by three normal distributions, the Euro and British pound by two normal distributions and the US Dollar currency pairs by one. The question arises of whether a significant exogenous event exists to account for such a pattern. To explain these results the data is partitioned pre- and post- the Japan earthquake of 2011 and the Brexit vote of 2016 and a Chow test applied to determine if more information can be derived from the sub periods than the whole. The null hypothesis of no breaks at the specified break point was not rejected and therefore concludes that more information cannot be derived from the sub periods than from the whole period in entirety. The results are statistical as expected.

5.4 Conclusions

This chapter reviewed the stochastic price processes that represented the systemic constituents of the foreign exchange market. Critiquing the systemic elements of the foreign exchange (FX) spot price: order flow, bid-ask spread and triangulation, Kon (1984) proposed that the constituent pricing factors are normally distributed but when operating as a system the resulting distribution is leptokurtic and positively skewed. The discrete mixture of normal distributions advances an explanation for the resulting distribution, attributed to the mean and variance shift parameters. These parameters arise when the pricing elements are separated out into their normally distributed constituents and differenced. The resulting shift parameters can then be applied to transform the leptokurtic, positivity skewed distribution to normal.

The most common assumption is that returns are multivariate normal with parameters that are stationary over time. The assessment of the normality hypothesis on the foreign exchange daily returns revealed that the D'Agostino-Pearson test rejected the null hypothesis of normality, further the distributions are in fact negatively skewed and leptokurtic with fatter-tails than a normal distribution with the same mean and variance attributed to a shift in the parameters over time.

A possible explanation for the observed skewness and kurtosis of the returns involves the model specification in which the true generating process is a mixture of normal distributions. Partitioning the data into sub-periods accounted for the shifts in the parameters, reducing the frequency the stationary normal null hypothesis is rejected. The thesis estimated the N normal distributions attributable to each currency pair to identify the optimal number of normal distributions to ensure normality of the return distribution. At the 5% level of significance, the sample described the USD/JPY, EUR/JPY and GBP/JPY by three normal distributions, two normal distributions for EUR/GBP with EUR/USD and GBP/USD described by one normal distribution.

The parameter estimates and stationarity test for the mixture of one, two and three normal distribution models for each of the currency pairs contained at least one mean parameter estimate that was negative. This negative parameter is consistent with the Monday effect. Monday returns usually exhibit higher variance than other days of the week while negative mean estimates are associated with a lower variance than the variance of the distributions of positive mean estimates. The inference that this thesis draws is that the true mixture of normal distributions is more complex than a simple partition of the data by year *and* day of the week. Accepting this limitation, the models identified demonstrate that the distribution of daily rates of return is comprised of the composite distributions of the systemic elements that make up the foreign exchange market. The Black and Scholes (1973) and R. C. Merton (1973) option pricing model is based on a single normal distribution and a geometric Brownian motion process of the underlying asset's price, where normality is crucial to models of financial theory. The option pricing formula can be improved by applying the shift parameter normal transform for the Kon (1984) mixture of $N = 1, 2$ or 3 normal distributions that arise from the systemic elements that make up the FX option price.

Chapter 6

FX Option Pricing and Forecasting

The application of option pricing theory can be adapted to price options on processes suggested by empirical literature. One such process is the discrete mixture of normal distributions model proposed by Kon (1984). The model might be formulated as a stochastic process whose drift parameter and volatility are time dependent and shift along a finite number of values to be determined from real data. Once the process is specified the Black and Scholes (1973) argument could be adapted to price the option. The alternative option pricing formula in this case would be the expected value of the standard Black and Scholes (1973) formula conditioned on the distribution of the volatility parameter where the values for variance σ^2 would be drawn from a discrete set.

Geometric Brownian motion is the stock diffusion model on which the Black and Scholes (1973) equation is based. While a very good first approximation for price changes, it can be improved significantly. A critical factor in the original Black and Scholes (1973) analysis is the failure of stock price returns to meet the precise description of the stochastic process governing the behaviour of the underlying asset.

In an attempt to improve upon the original Black and Scholes (1973) and R. C. Merton (1973) (BSM) equation, the assumption that the asset price follows a geometric Brownian motion will be relaxed. Naturally, any price process suggested for replacing or modifying the geometric Brownian motion must result in a formula that can be utilised by option traders to improve on the performance of the embedded BSM equation. The Kon (1984) discrete mixture of normal distributions stochastic model will be applied to formulate an alternate approach that is more aligned to the geometric Brownian motion price process when applied to foreign exchange. An alternative FX option pricing model will be derived from enabling the Black and Scholes (1973) and R. C. Merton (1973) methodology pertaining to the Kon (1984) price process.

Risk reversals derived from the alternative FX option pricing model will be used to forecast the spot market. This forecasting performance will be compared to the BSM market derived risk reversals. The premise being that an improved model should afford improved forecasting abilities.

6.1 Properties of Option Pricing Models

An American call option gives the owner the right to purchase a share of stock at a given exercise price on or before a given date and is issued by an individual or financial institution. An American put option gives its owner the right to sell a share of stock at a given exercise price on or before a given date. A European option has the same terms as its American counterpart except that it cannot be exercised before the last date of the contract. P. Samuelson (1965) demonstrated that the two types of contracts may not have the same value since the contracts may differ with respect to other provisions.

This paper will focus on European options.

6.1.1 Properties of the Call Option Price Formula

An insight into the properties of the Black and Scholes (1973) formula is gained by determining how the model performs when the parameters take on extreme values and its adherence to the boundary conditions: J. C. Hull (2012)

i. What happens when the stock price S becomes much larger than the strike price K: the option becomes very similar to a forward contract with the probability the option will be exercised tending towards unity.

As the stock price $S \longrightarrow \infty$ the cumulative probability distribution $N(d_1) \longrightarrow 1$ and $N(d_2) \longrightarrow 1$, so that the call option price f_c approaches the expected price $S - Ke^{-rT}$.

ii. What happens when the volatility $\sigma \longrightarrow 0$: the stock tends towards being riskless.

This case expects the option to behave like a risk-free bond whose price grows at the risk-free rate r. Thus at time T the call option payoff for the riskless stock is

$$max(Se^{rT} - K, 0)\,, \tag{6.1}$$

thus the present discounted value of the call option becomes

$$e^{-rT}max(Se^{rT} - K, 0) = max(S - Ke^{-rT}, 0)\,. \tag{6.2}$$

To show that the Black and Scholes (1973) formula is consistent with Equation (6.2) consider the case when $S > Ke^{-rT}$. Rewriting this inequality as $\ln(S/K)+rT > 0$, when $\sigma \longrightarrow 0$, d_1 and $d_2 \longrightarrow +\infty$. Thus $N(d_1)$ and $N(d_2) \longrightarrow 1$ and giving $f_c = S - Ke^{-rT}$ in the limit. Conversely when $S < Ke^{-rT}$, then as $\sigma \longrightarrow 0$, d_1 and $d_2 \longrightarrow -\infty$. Thus $N(d_1)$ and $N(d_2) \longrightarrow 0$ which gives $f_c = 0$ for the call option price.

iii. What happens when the volatility $\sigma \longrightarrow \infty$: in the limit the option price equals the stock price, $f_c = S$, thus the strike price K becomes irrelevant.

In the limit as $\sigma \longrightarrow \infty$, $d_1 \longrightarrow +\infty$ and $d_2 \longrightarrow -\infty$, thus $N(d_1) \longrightarrow 1$ and $N(d_2) \longrightarrow 0$. When the stock price S is fundamentally uncertain the strike price K becomes unimportant: why purchase and option to acquire the stock at the strike price K when you can buy it directly from the market at the stock price S.

Therefore, even under extreme circumstances the Black and Scholes (1973) model adheres to the option boundary conditions.

6.2 Literature Review

6.2.1 Option Valuation Theory

A central problem of modern finance is that of valuing the claims to assets. Modigliani and Miller (1958) stated that at equilibrium, packages of financial claims which are in essence equivalent, must command the same price. As a consequence the aggregate value of a claim is independent of the type of claim issued. This argument can be applied to a specialised form of financial claim in evaluating options. The seminal work in this area is the Black and Scholes (1973) option pricing model which only depends on observable variables.

Cox and Ross (1976) presented an option valuation framework which illustrates the structure of hedging arguments to obtain a valuation formula for vanilla options. This framework can be adapted for the valuation of FX options, it is stated as:

i. Choose a particular stochastic process to govern the price moment of the FX market, spot price S.

ii. Take an instrument whose value is dependent on S and assume a regular price function $f(S, t)$ exists.

iii. Assuming the price process and option function $f(S, t)$ are well behaved, derive the differential equation for the option value, df.

iv. Note that the drift and variance parameters of the option price df depend on the unknown function $f(S, t)$ and known values of S and t.

v. Assume short selling and the existence of a riskless asset that earns an instantaneous rate r. Assume no arbitrage.

vi. For Poisson price processes assume the jump amplitude is a non-random function.

vii. Formulate a differential-difference equation for the option price.

viii. Use the terms of the option to set boundary conditions in solving the differential-difference equation.

The Cox and Ross (1976) framework requires that the random differential movement of S be written as

$$dS = \mu(S,t)\,dt + \sigma(S,t)\,dx\,, \tag{6.3}$$

where $\mu(S,t)$ and $\sigma(S,t)$ are functions of the current state of the world, and that the option process df is expressed as

$$df = \mu(f,t)\,dt + \sigma(f,t)\,dx\,. \tag{6.4}$$

Cox and Ross (1976) use the existence of a hedged portfolio of the stock, S, and the option, $f(S,t)$, to define the relationship

$$\alpha_S\,\sigma_S\,(dx_S/S) + \alpha_f\,\sigma_f\,(dx_S/f) = 0\,, \tag{6.5}$$

where the dependence on the t is dropped for simplicity. This simplifies to

$$\alpha_S(\sigma_S/S) + \alpha_f(\sigma_f/f) = 0\,, \tag{6.6}$$

where α_S and α_f are the portfolio weights in the stock and option respectively. The hedged portfolio is riskless and must have a rate of return of

$$\alpha_S(\mu_S/S) + \alpha_f(\mu_f/f) = (\alpha_S + \alpha_f)r\,. \tag{6.7}$$

Separating out the risky and riskless components gives

$$\alpha_S\left(\frac{dS}{S}\right) + \alpha_f\left(\frac{df}{f}\right) = (\alpha_S + \alpha_f)r\,, \tag{6.8}$$

where the total return for the stock holding plus the total return from the option holding must equal the risk-free return. From Equations (6.5) and (6.7) the fundamental option valuation equation gives

$$\frac{(\mu_f - rf)}{\sigma_f} = \frac{(\mu_S - rS)}{\sigma_S}\,, \tag{6.9}$$

where the risk premium divided by the scale of risk has to be the same for the stock and the option. It can be argued that the uniqueness of the solution $f(S,t)$ and the independence of the hedging argument from any presumption about the risk preferences of the investor imply that the expected return on the stock and the option can be set equal to the risk-free interest rate r.

6.2.2 Rational Option Pricing Theory

Bachelier (1900) deduced an option pricing formula based on the assumption that stock prices follow a Brownian motion with zero drift. In contributing to the theory R. C. Merton (1976) extrapolated a set of restrictions necessary for an option pricing formula to be consistent with a rational pricing theory based on the assumption that investors prefer more to less.

R. C. Merton (1976) noted that in perfect markets with no transaction costs and the ability to borrow and short-sell without restriction, the existence of a dominated security: given securities A and B, A dominates B if the return on security A exceeds the return on security B for some possible states of the world, and will be at least as large as B in all possible states of the world, would be equivalent to the existence of an arbitrage situation. However, it is possible to have dominated securities exist without arbitrage in imperfect markets. If one assumes symmetric market rationality and that investors prefer more wealth to less, then any investor willing to purchase security B would prefer to purchase security A.

Assumption 1: A necessary condition for a rational option pricing theory is that the option be priced such that it is neither a dominant nor a dominated security

Let $f(S,T;K)$ denote the value of a call option with stock price S, time to maturity T, and exercise price K. From the definition of a call option and limited liability, where

$$f(S,T;K) \geq 0\,, \tag{6.10}$$

then it follows from *Assumption 1*, that

$$f(S,T_1;K) \geq f(S,T_2;K) \quad \text{if} \quad T_1 > T_2\,. \tag{6.11}$$

Further, if one option has a larger exercise price than the other, it must satisfy

$$f(S,T;K_1) \leq f(S,T;K_2) \quad \text{if} \quad K_1 > K_2\,. \tag{6.12}$$

Because the common stock is equivalent to a perpetual ($T = \infty$) call option with a zero exercise price ($K = 0$), it follows

$$S \geq f(S,T;K)\,. \tag{6.13}$$

Let $P(T)$ be the price of a riskless discounted bond which pays one dollar T years from now. If current and future interest rates are positive, then

$$1 = P(0) > P(T_1) > P(T_2) > ... > P(T_n) \quad \text{for} \quad 0 < T_1 < T_2 < \cdots < T_n\,. \tag{6.14}$$

From *Assumption 1* and Equation (6.10) it follows that if the exercise price of a European call is K and if no dividend payments are made over the life of the option, then

$$f(S,T;K) \geq max[S - KP(T), 0]\,, \tag{6.15}$$

adhering to the boundary conditions of a call option. The boundary condition $max[S-K,0]$ is referred to as the intrinsic value of the option. The option must always sell for at least its intrinsic value. If the intrinsic value holds, then the value of an option with a large time to maturity ($T=\infty$) must equal the value of the common stock. Conversely, finite time to maturity option prices must be a function of the price of a riskless discounted bond $P(T)$. If this were not the case then for some sufficiently small $P(T)$ the intrinsic value conditions would be violated.

R. C. Merton (1973) argued that the use of interest rates was implied in the formula by using the exercise price as a variable instead of the present value of the exercise price. An argument for the reasonableness of this result comes from recognising that a European call option is equivalent to a long position in the common stock leveraged by a limited-liability discount loan, where the borrower promises to pay K dollars at the end of T periods. In the event of default the borrower is only liable to the extent of the value of the common stock at that time. If the present value of such a loan is a decreasing function of the interest rate, then for a given stock price, the option price will be an increasing function of the interest rate.

R. C. Merton (1973) concluded that the rationally determined option price is a non-decreasing function of the riskiness of its associated common stock. The more uncertain one is about the outcomes on the common stock the more valuable is the option.

6.2.3 Alternative Option Diffusion Processes

Black and Scholes (1973) assumed the value of the stock follows a geometric Brownian motion through time which produces a log-normal distribution for stock prices between any two points in time. As a consequence a portfolio consisting of stock and any option written on it will be perfectly correlated. Combining this portfolio with borrowing or lending at the risk-free rate a position in one will be a perfect substitute for the other. In this way the option is covered by riskless bonds and the stock. Thus if the value of the stock is known, then one can value the option. The critical factor in this argument is the precise description of the stochastic process governing the behaviour of the underlying asset. The characteristics of this process determine the exact nature of the equivalence between packages of financial claims. In considering alternative forms of stochastic processes governing stock prices Cox and Ross (1976) developed an approach to the option valuation problem that connects it to the underlying stochastic process.

The basic assumption employed by Black and Scholes (1973) was that the stock value followed a log-normal process

$$\frac{dS}{S} = \mu\, dt + \sigma\, dz\,, \tag{6.16}$$

where S is the value of the stock with drift term μ, variance σ^2 and Wiener process z. This is a short-hand notation for the stochastic process were S_t is the value of the stock at time t and the percentage change in the value in the next instant from t to $t + dt$ is

$$\frac{dS}{S} = \frac{S_{t+dt} - S_t}{S_t} \,. \tag{6.17}$$

The percentage change is made up of two components, a drift term μdt which is certain when viewed from time t and a normally distributed stochastic term σdz. The stochastic term is independent of its values in previous periods and has a zero mean and variance $\sigma^2 dt$. The percentage change in stock value from t to $t + dt$ is normally distributed with mean μdt and variance $\sigma^2 dt$. At the limit of dt, S_{t+dt} will not differ much from S_t. This is the fundamental diffusion process and represents a random walk around a trend term in the short run and offers no surprises.

The original Black and Scholes (1973) approach yielded the differential equation of the form stated in Equation (G.10) on page 197. When considering alternative diffusion processes look at the Linear Price Variance and Constant Price Variance Processes. For the Linear Price Variance Process the differential equation becomes

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S\frac{\partial^2 f}{\partial S^2} = rf \,. \tag{6.18}$$

The density of the limiting diffusion for the birth and death process is known: Feller (1951), therefore the risk-neutral method can be applied and the expectation of $max(S(T) - K, 0)$ discounted to time t can be taken to obtain the valuation formula

$$f(S,t) = S\sum_{n=0}^{\infty} \frac{(n+1)e^{-y}y^n G(n+2, \theta K)}{\Gamma[n+2]} - Ke^{-r(T-t)}\sum_{n=0}^{infty} \frac{e^{-y}y^{n+1}G(n+1, \theta K)}{\Gamma[n+2]} \,, \tag{6.19}$$

where

$$\begin{aligned} \theta &= \frac{2r}{\sigma^2[e^{r(T-t)} - 1]} \,, \\ y &= \theta S e^{(T-t)} \,, \\ G(m,x) &= [\Gamma(m)]^{-1} \int_x^{\infty} e^{-z} z^{m-1}\, dz \,. \end{aligned} \tag{6.20}$$

For the Constant Price Variance Process the differential equation now becomes

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial S^2} = rf \,. \tag{6.21}$$

As with the previous process the density of the diffusion is known generating the following valuation formula

$$f(S,t) = \left(S - Ke^{-r(T-t)}\right) N(y_1) + \left(S + Ke^{-r(T-t)}\right) N(y_2) + v[n(y_1) - n(y_2)]\,, \tag{6.22}$$

where $N(\cdot)$ is the cumulative unit normal distribution function, $n()$ is the unit normal density function and

$$\begin{aligned} v &= \sigma\sqrt{\left(\frac{1 - e^{-2r(T-t)}}{2r}\right)}\,, \\ y_1 &= \frac{S - Ke^{-r(T-t)}}{v}\,, \\ y_2 &= \frac{-S - Ke^{-r(T-t)}}{v}\,. \end{aligned} \tag{6.23}$$

The diffusion process is only one of two general classes of continuous time stochastic processes. The other type of stochastic process in continuous time is the jump process.

Resulting Jump Processes

A simple jump process can be written as

$$\begin{aligned} \frac{dS}{S} &= \mu dt + (k-1)d\pi\,, \\ \frac{dS}{S} &= \begin{cases} (k-1), & \text{with probability } \lambda\, dt\,, \\ 0 & \text{with probability } 1 - \lambda\, dt\,, \end{cases} \end{aligned} \tag{6.24}$$

where π is a continuous time Poisson process, λ is the intensity of the process and $k-1$ is the jump amplitude. Equation (6.24) is a shorthand notation for the stochastic process that governs the percentage change in the value of the stock on the interval from t to $t+dt$. This percentage change is composed of a drift term μdt and a term $d\pi$, which with a probability λdt will jump the percentage stock change to the random amplitude $k-1$ and with a probability of $1-\lambda dt$ it will do nothing. An interpretation of λdt is the instantaneous probability of receiving a packet of information that will cause S to jump.

In contrast to the diffusion process, the jump process follows a deterministic movement upon which is superimposed discrete jumps. Formally, a jump process has a discontinuous sample path with probability one whereas the diffusion process has a continuous sample path with probability one.

Because of the jumps in value the Black and Scholes (1973) analysis for valuing options does not carry directly over to Equation (6.24).

For a stock whose return is given by

$$dS = \mu S\, dt + dq\,, \tag{6.25}$$

where dq are the increments of the pure jump process q, given by

$$dq = \begin{cases} (k-1), & \text{with probability } \lambda\, dt\,, \\ 0, & \text{with probability } 1-\lambda\, dt\,, \end{cases} \tag{6.26}$$

the option $f(S,t)$ follows the dependent process

$$df = \begin{cases} f(S+k-1,t) - f(S,t), & \text{with probability } \lambda S\, dt\,, \\ \\ \frac{\partial f}{\partial t} dt + \mu \frac{\partial f}{\partial S} dt, & \text{with probability } 1-\lambda S\, dt \end{cases} \tag{6.27}$$

where λ is an arbitrary function. By constructing the fundamental option valuation equation for this option process, Cox and Ross (1976) obtain the following difference-differential equation

$$\mu \frac{\partial f}{\partial S} + \left[\frac{\mu - rS}{1-k}\right] f(S+k-1,t) + \left[\frac{r[k-1+S]-\mu}{1-k}\right] f(S,t) + \frac{\partial f}{\partial t} = 0\,, \tag{6.28}$$

where μ and k are functions of S and t. Note that Equation (6.28) is independent of the jump process intensity λ. When the hedge position depends only on the jump size then the intensity plays no role in the valuation.

Merton's Mixed Processes

R. C. Merton (1976) examined a jump process with a drift term, although there is no closed-form solution the underlying stochastic process is stated in terms of the return dS/S rather than the price increment dS. The return process used by R. C. Merton (1976) is

$$\frac{dS}{S} = (\mu - \lambda k)\, dt + \sigma\, dz + dq\,, \tag{6.29}$$

where the pure jump process is given by

$$dq = \begin{cases} Y-1, & \text{with probability } \lambda\, dt\,, \\ 0, & \text{with probability } 1-\lambda\, dt\,. \end{cases} \tag{6.30}$$

Using this variation R. C. Merton (1976) arrived at the following difference-differential equation for the option price $f(S,t)$

$$\frac{\partial f}{\partial t} + (r-\lambda k)S\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + \lambda\epsilon\{f(SY,t) - f(S,t)\} = rf\,, \tag{6.31}$$

subject to the boundary conditions

$$\begin{aligned} &f(0,t) = 0 \\ \text{and} \quad & \\ &f(S,T) = max(S-K,0)\,. \end{aligned} \tag{6.32}$$

While a closed-form solution to Equation (6.31) cannot be written down without further specification, a partial solution can be. Define $W(S,t;K,r,\sigma^2)$ to be the Black and Scholes (1973) option pricing formula for the non-jump case in Equation (G.10). Define the random variable X_n to have the same distribution as the product of n i.i.d. random variables, each identically distributed to the random variable Y where $X_0 = 0$. Define ϵ_n to be the expectation operator over the distribution of X_n. Then the solution to Equation (6.31) when the current stock price is S can be written as

$$f(S,t) = \sum_{n=0}^{\infty} \frac{e^{-\lambda(T-t)}\lambda^n (T-t)^n}{n!} \left[\epsilon_n\{W(SX_n e^{-\lambda k(T-t)}, t; K, \sigma^2), r\}\right]. \tag{6.33}$$

6.2.4 Alternatives to the Black-Scholes Model

The Black and Scholes (1973) closed-form option pricing model is based on the assumption that the price of the underlying asset S follows a log-normal process with an expected return μ and a constant volatility σ

$$\frac{dS}{S} = \mu\, dt + \sigma\, dz\,, \tag{6.34}$$

where S is the stock price and dS/S follows a geometric Brownian motion with expected rate of return μ and variance σ^2.

The drawback to this model is that it is based on the simplistic assumptions of constant volatility and a normal distribution function for the underlying asset return. The limitations of the model are evidenced by the discrepancies observed between market and theoretical prices. These discrepancies are pronounced in the different implied volatilities according to the exercise price (smile and skew) and maturities (term structure): Rubinstein (1994) and Dumas et al. (1998).

Alternatives to the Black and Scholes (1973) model were developed to account for the implied volatility smiles and for the skewness and kurtosis in the distribution of the returns. These alternatives are divided into three categories. The first consists of models with a jump diffusion process: R. C. Merton (1976). The second category comprises of stochastic volatility models: J. Hull and White (1987), Stein and Stein (1991) and Heston (1993). The third category forms the local volatility model developed: Dupire et al. (1994), Derman and Kani (1994) and Rubinstein (1994).

The following sections describe each of these categories, highlighting the benefits and limitations of each.

Jump Diffusion Models

R. C. Merton (1976) introduced a jump component following a Poisson process to the Black and Scholes (1973) diffusion process. The intuition that markets receive new information in a discrete manner and this information causes them to adjust the underlying asset price accordingly. The jump component is representative of actual market conditions by accounting for outliers and the asymmetry of the return distribution. The diffusion process is given by

$$\frac{dS}{S} = \mu dt + \sigma dz + dq\,, \tag{6.35}$$

where dq is the jump component and follows a Poisson process. Ball et al. (1985), Das and Uppal (2004) and Das (2002) tested the model after estimating values for its parameters and found the jump component to be significant. The model benefits from being able to generate several different volatility smile and skew shapes according to what is observed in the markets. For instance, using a jump process with a negative average presents the sharp skew seen for short-term maturities. Das and Foresi (1996), Bates (1996) and Bakshi et al. (1997) illustrated the importance of the jump component for pricing very short-term options. Bakshi et al. (1997) compared several models and noted that jumps are essential for modelling the asset diffusion process when pricing short-term options even if using a stochastic volatility model. Bakshi et al. (1997) concluded that models with jumps better account for the skewness and kurtosis in the distribution of returns. However models with jumps are difficult to put into practice. Such models do not offer a simple equation to calculate option prices.

Stochastic Volatility Models

Although the Black and Scholes (1973) formula is successful in explaining stock option prices, it does have known biases: Rubinstein (1985). Its performance also is substantially worse on foreign currency options: Melino and Turnbull (1990) and Melino and Turnbull (1991). Subsequent pricing models aim to ease one or more of the BSM assumptions, *see* Section G.1.1 on page 194, with the presumption of a constant volatility across strike prices and maturity being the most problematic.

In these models the underlying asset volatility is considered stochastic and its movements are represented by a diffusion process. The diffusion process in stochastic volatility models is still considered log-normal, *see* Equation (6.34), but in this case the volatility σ is not constant. The volatility also follows a stochastic process which can be correlated with the underlying asset diffusion process. Therefore a two-equation system is required to model the underlying movements. J. Hull and White (1987) suggested that the variance equation follows a log-normal process given by

$$\frac{d\sigma^2}{\sigma^2} = \delta\, dt + \xi\, dz\,, \tag{6.36}$$

where δ and ξ represent the drift and volatility of the volatility. Solving this two equation system with partial derivatives requires complex numerical methods. Given that the volatility can neither increase nor decrease indefinitely, Stein and Stein (1991) developed a mean reversion process for the volatility diffusions, given as

$$d\sigma = \delta(\theta - \sigma)\, dt + \xi\, dz\,, \tag{6.37}$$

where θ is the mean reversion level. In developing an option pricing model Stein and Stein (1991) assumed that the volatility was not correlated with the underlying asset. This assumption does not allow for the skewness in the return distribution. Conversely Bakshi et al. (1997) noted that in stochastic volatility models the correlation between the volatility and the underlying asset determines the level of skewness while the volatility of the volatility determines the level of kurtosis in the distribution of the asset returns.

Bakshi et al. (1997) added a jump component to stochastic volatility models to merge the best elements of the two models. They compared stochastic volatility models with and without jumps. The jump element improved the model for short-term options and accounted for the stochastic volatility. However the model required improbable values for correlation and the volatility of volatility to represent the skewness and kurtosis in the distribution of the returns.

Heston (1993) presented a stochastic volatility model that was not based on the Black-Scholes formula. It provided a closed-form solution for the price of a European call option when the spot asset is correlated with volatility. The model can be adapted and applied to currency options by incorporating stochastic interest rates.

Heston (1993) relaxed the Stein and Stein (1991) model by allowing volatility to be correlated to the underlying asset return, the variance follows a mean reversion function given by

$$d\sigma^2 = \delta(\theta - \sigma^2)\, dt + \xi\, dz\,. \tag{6.38}$$

The advantage of stochastic volatility models is that they give a different smile and skew shape depending on the parameters used for the volatility process and the level of correlation between the volatility and the underlying asset return. Nonetheless, these models are difficult to use and cannot be employed for dynamic hedging. Bates (1996) and Bakshi et al. (1997) concluded that any representation of the underlying asset price diffusion must incorporate both a stochastic volatility and a jump diffusion process.

Local Volatility Models

Stochastic volatility models can reproduce the implied volatility curve shape typically seen in the markets, but cannot easily be calibrated with any implied volatility surface. Research into a version that can be calibrated from the observed implied volatility surface led to the development of local volatility models. These models provide a simple way to price options using implied trees. There are two types of local volatility models: deterministic and stochastic local volatility models.

i. *Deterministic Local Volatility Models*: Dupire et al. (1994) initiated research into local volatility models, removing the constant volatility assumption and introducing the local volatility theory whereby the instantaneous volatility is considered to be a deterministic function of time and the underlying asset, given as

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma(t, S_t)\, dz_t\,. \tag{6.39}$$

The local volatility $\sigma_L(t, S_t)$ is equal to the instantaneous volatility $\sigma(t, S_t)$ at a future time t and corresponding underlying asset value S_t, pricing all observed options in a consistent manner. The local volatility function can be determined from a market price surface $C(T, K)$ of standard European options with different exercise prices and maturities, leading to the Dupire formula

$$\sigma_L^2(t, S_t | t = T, S_t = K) = 2\, \frac{\frac{\partial C}{\partial T} + (r-q)\, K\, \frac{\partial C}{\partial K} + qC}{K^2\, \frac{\partial^2 C}{\partial K^2}}\,. \tag{6.40}$$

Once the local volatility function has been determined the model calculates future underlying asset prices. Therefore an option can be priced using this diffusion process which will be consistent with all liquid options on the same underlying asset.

Traders use the local volatility approach because it is easy to use and retains the completeness of the Black and Scholes (1973) model. Conversely Andersen and Andreasen (2000) criticised this approach, claiming that movements in the smile in a deterministic local volatility model lead to unstationary implied volatilities, implying that the volatility skew will disappear in the future, contradicting empirical observations. Andersen and Andreasen (2000) extended the deterministic local volatility models to account for jumps in the underlying asset diffusion process. This brings together the jump process for modelling steep short-term skews and the local volatility to ensure the model is in line with market option prices.

ii. *Stochastic Local Volatility Models*: Due to the dynamic hedging limitation of deterministic local volatility models researchers sought to enhance them. Alexander and Nogueira (2004) pointed out that the deterministic local volatility assumption implies that the instantaneous volatility is also deterministic, contradicting empirical studies of stochastic volatility models. Therefore Alexander and Nogueira (2004) developed a stochastic local volatility model to merge the two model types. Alexander and Nogueira (2004) noted that the local volatility surface may be a deterministic function of t and S where the surface changes with time in a stochastic manner. The equation for the underlying diffusion process is

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\left(t, S_t : \nu_1(t), \ldots, \nu_n(t)\right) dz_t\,, \tag{6.41}$$

where $(\nu_1(t), \ldots, \nu_n(t))$ are those of a deterministic local volatility model: stochastic functions correlated with each other and with the underlying asset. Therefore the instantaneous volatility $\sigma\left(t, S_t, \nu(t)\right)$ is also stochastic.

6.2.5 Black-Scholes and the Foreign Exchange Option Pricing Model

Since almost all corporate liabilities can be viewed as combinations of options, the formula and the analysis that lead to a theoretical valuation formula for options is also applicable to corporate liabilities such as common stock, corporate bonds, and warrants. In particular the formula can be used to derive the discount that should be applied to a corporate bond because of the possibility of default.

Previous work on the valuation of options has been expressed in terms of warrants: a security issued by a company giving its owner the right to purchase a share of stock at a given exercise price on or before a given date: Sprenkle (1961), A. James Boness (1964) and P. Samuelson (1965).

The Sprenkle (1961) formula for the value of an option can be written as

$$f(S,t) = \iota S N(b_1) - \iota^* K N(b_2)\,,$$

$$\text{where} \quad b_1 = \frac{\ln(\iota S/K) + \frac{1}{2}\sigma^2(t^* - t)}{\sigma\sqrt{(t^* - t)}} \tag{6.42}$$

$$\text{and} \quad b_2 = \frac{\ln(\iota S/K) - \frac{1}{2}\sigma^2(t^* - t)}{\sigma\sqrt{(t^* - t)}}\,,$$

where S is the stock price, K the strike price, t^* is the maturity date and t is the current date, σ^2 is the variance rate of the return on the stock, $N(b)$ is the cumulative normal density function and ι and ι^* are unknown parameters.

Sprenkle (1961) defined ι as the ratio of the expected value of the stock price at the time the warrant matures to the current stock price and ι^* as a discount factor that depends on the risk of the stock. In attempting to estimate ι and ι^* empirically Sprenkle (1961) found that it was not possible to do so.

P. Samuelson (1965) assumed the distribution of the stock values when the warrant matures is log-normal and takes the expected values: proposed unknown parameters α: rate of expected return on the stock and β: the rate of expected return on the warrant or the discount rate to be applied to the warrant. Cutting it off at the exercise price, P. Samuelson (1965) discounted this expected value at the rate β. Unfortunately there was no model for pricing securities under this method.

P. Samuelson and Merton (1969) noted that discounting the expected value of the distribution of possible values of the warrant when it is exercised is not an appropriate procedure. P. Samuelson and Merton (1969) advocated treating the option price as a function of the stock price, acknowledging that the discount rates are determined in part by the willingness of investors to hold outstanding amounts of both the stock and the option. However they failed to recognise that investors must hold other assets as well, so that the risk of an option or stock that affects the discount rate is only that part of the risk that cannot be diversified away.

One of the concepts Black and Scholes (1973) used in developing their model was that of Thorp and Kassouf (1967) who obtained an empirical valuation for warrants by fitting a curve to actual warrant prices. Thorp and Kassouf (1967) then used this formula to calculate the ratio of shares to options needed to create a hedge position. However Thorp and Kassouf (1967) did not consider that at equilibrium the expected return on a hedged position must be equal to the return on a riskless asset. Black and Scholes (1973) used this equilibrium condition to derive a theoretical valuation formula.

Black and Scholes (1973) noted that if options are correctly priced it is not possible to make a profit by creating portfolios of long and short positions in options and their underlying stocks. The option will depend only on the price of the stock and time and on variables that are taken to be known constants. Under the assumptions on page 194 the stock price follows a continuous random walk and the return has a constant variance rate, the covariance between the return on the equity: the net position between the cost of the stock and income from the sold option, and the return on the stock will be zero. Thus the risk in the hedged position is zero if the short position in the option is adjusted continuously. Note that the direction of the change in the equity value is independent of the direction of the change in the stock price. Using this principle Black and Scholes (1973) derived an option pricing formula given in Equation (G.14) on page 198. R. C. Merton (1973) has shown that the option value as given by this equation increases continuously as any one of σ^2, r or t increases. In each case, it approaches a maximum value equal to the stock price.

The partial derivative $\frac{\partial f}{\partial S}$ in Equation (G.10) on page 197 is of interest, because it determines the ratio of shares of stock to options in the hedged position as in Equation (G.6) on page 196. Taking the partial derivative of Equation (G.10), and simplifying, gives

$$\frac{\partial f}{\partial S} = N(d_1)\,, \tag{6.43}$$

where (d_1) is as defined in Equation (G.14). It is clear that $S\frac{\partial f}{\partial S}$ in Equation (G.10) is always greater than one. This shows that the option is always more volatile than the stock.

Valuation of Currency Options

A European call option on a foreign exchange is defined as the right, but not the obligation, to buy one unit of the currency on a predetermined date at a predetermined exchange rate. The prices of foreign currency options are important in determining the values of other financial contracts. Feiger and Jacquillat (1979) considered the currency option bond: the bond holder can choose in which currency the coupons and principle are paid out according to a pre-determined exchange rate, noting that this was equivalent to a single currency bond plus a foreign currency option, giving

$$S = B + cp\,, \tag{6.44}$$

where S is the bond price paying either \$1 or £p at time T, B the price of a pure discount bond paying \$1 at time T and c is the price of a European call to purchase \$1 for a dollar price of $\frac{1}{p}$ at time T.

Investors can utilise currency options by combining a short position in a currency with a call option, limiting the downside risk while benefiting from favourable exchange rate movements. Foreign currency options are also employed by corporations uncertain whether it will have a long position in a currency. Feiger and Jacquillat (1979) noted in such circumstances the combination of a forward contract to sell currency at time T and a call option to buy it at time T provides a hedge not available in forward contracts alone.

Feiger and Jacquillat (1979) developed a valuation model for two-currency, currency option bonds, assuming a joint stochastic process for the exchange rate, domestic and foreign interest rates. Stulz (1982) developed a series of analytical formulas for European put and call options on the minimum or maximum of two risky assets. These can be applied to value currency options. Under the assumption that a stochastic process for just one variable produces a simpler valuation formula than proposed by Feiger and Jacquillat (1979), Biger and Hull (1983) derived a valuation formula for European put and call foreign exchange options using the Black-Scholes methodology. Biger and Hull (1983) valued European put and call options on a foreign currency under the following assumptions:

i. The price of one unit of foreign currency follows a geometric Brownian motion.

ii. The foreign exchange market operates continuously with no transaction costs or taxes.

iii. The risk-free interest rates in both the foreign and domestic country are constant during the life of the option.

Given the choice foreign currency investors would always choose to invest in short-term risk-free foreign currency bonds in preference to holding the foreign currency in some non-interest-bearing account. Assume a holding of a foreign currency gives a return equal to the foreign risk-free rate; therefore valuing an option on a foreign currency can be viewed as being the same problem as valuing an option on a stock paying a continuous dividend. Black and Scholes (1973) assumed that no dividends are paid on the stock during the life of the option, thus their model cannot be directly applied to value an option on a foreign currency. R. C. Merton (1973) and Smith Jr (1976) address this by making the assumption that the dividend yield, q, is constant. Constructing a riskless hedge and applying the dividend yield q, the Black-Scholes foreign exchange valuation formula becomes

$$c = Se^{-qT}N(d_1) - Ke^{-r_dT}N(d_2)\,,$$

$$p = Ke^{-r_dT}N(-d_2) - Se^{-qT}N(-d_1)\,,$$

$$\text{where}\quad d_1 = \frac{\ln(S/K) + (r_d - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \tag{6.45}$$

$$\text{and}\quad d_2 = \frac{\ln(S/K) + (r_d - q - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\,,$$

where c is a European call and p a European put option to purchase or sell one unit of the foreign currency, S is the spot price of one unit of the foreign currency, σ^2 the instantaneous variance of the return on a foreign currency holding, K the exercise price, T the time to maturity, r_d is the domestic country risk-free rate and q is the dividend yield.

The deliverable instrument of an FX option is a fixed amount of underlying foreign currency. In the standard Black and Scholes (1973) option-pricing model, the underlying deliverable instrument is a non-dividend-paying stock. The difference between the two underlying instruments is readily seen when comparing their equilibrium forward prices. When interest rates are constant, as in the Black-Scholes assumptions, arbitrage ensures that the forward price of the stock commands a forward premium equal to the interest rate. But in the foreign currency markets, forward prices can involve either forward premiums or discounts. This is because the forward value of a currency is related to the ratio of the prices of riskless bonds traded in each country. The no-arbitrage condition for covered Interest Rate Parity requires the forward exchange premium to equal the interest rate differential, which may be either positive or negative. Thus both foreign and domestic interest rates play a role in the valuation of these forward contracts, and it is therefore logical to expect that such a role extends to options as well.

The forward rate plays a central role in the valuation of foreign currency options. Define F as the forward rate on the foreign currency for a contract with delivery date T. Interest Rate Parity theory implies

$$\ln(F/S) = (r_d - r_f)T\,. \tag{6.46}$$

If the foreign currency risk-free interest rate r_f is assumed to be constant, substituting Equation (6.46) into Equation (6.45) provides a valuation formula for a European call and put option written on the foreign currency when $q = r_f$ that is dependent on F rather than S, with K, σ, T and r_d the same as the Black-Scholes formula, given as

$$c = Fe^{-r_f T} N(d_1) - Ke^{-r_d T} N(d_2)\,,$$

$$p = Ke^{-r_d T} N(-d_2) - Fe^{-r_f T} N(-d_1)\,,$$

$$\text{where} \quad d_1 = \frac{\ln(F/K) + (\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \tag{6.47}$$

$$\text{and} \quad d_2 = \frac{\ln(F/K) - (\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\,.$$

This extension of the Black-Scholes approach enables the investor to form a riskless hedge by combining forward contracts with a short position in call options. At any given time, t, the portfolio is adjusted so that the ratio of forward contracts held to call options sold is $\partial c/\partial F_1$, where $F_1 = Fe^{-r(T-t)}$ and F is the forward rate at time t for a contract with delivery date T. An analysis analogous to that of Black and Scholes (1973) provides a differential equation relating c, p and F if it is assumed that:

i. F follows a geometric Brownian motion.

ii. The forward exchange market operates continuously with no transaction costs and no taxes.

iii. The domestic risk-free rate, r_d, is constant.

Thus, if it can be assumed that the forward rate follows a geometric Brownian motion it is not necessary to assume a constant foreign risk-free rate. When making the additional assumption that the foreign risk-free interest rate, r_f, is constant, F follows a geometric Brownian motion if and only if S does and σ_F the instantaneous standard deviation of F equals the instantaneous standard deviation of the return on a foreign currency holding. The result then becomes equivalent to Equation (6.47).

An Alternative Foreign Exchange Option Derivation

The modified Black-Scholes foreign exchange option pricing formula, Equations (6.47) can also be derived under the following assumptions:

i. The covariance of the returns from an option of the foreign currency and the returns from the international market portfolio is zero.

ii. The spot rate follows a geometric Brownian motion with instantaneous standard deviation σ.

iii. The international Sharpe-Lintner capital asset pricing model holds. The model converts the mean–variance Capital Asset Pricing Model (CAPM) model into a market-clearing asset-pricing model and assumes a risk-free rate.

Options on the foreign currency should be valued at their expected terminal value, discounted at the risk-free rate as

$$c = e^{-rT} \int_X^\infty (S_T - X)\, \mathrm{f}(S_t|S_0)\, dS_T$$

and

$$p = e^{-rT} \int_0^X (X - S_T)\, \mathrm{f}(S_t|S_0)\, dS_T\,, \tag{6.48}$$

where S_T is the price of the foreign currency at time T, S_0 is the price of the foreign currency at time 0, f is the probability distribution of S_T conditional on S_0 and assuming $\ln(S_T/S_0)$ is normally distributed with standard deviation $\sigma\sqrt{T}$.

The log-normal property of geometric Brownian motion has been widely used in stock option pricing models. This has an added appeal when applied to foreign currency. If the price of the foreign currency expressed in terms of the domestic currency is log-normal then the price of the domestic currency expressed in terms of the foreign currency is also log-normal.

Using integrals of the log-normal distribution derived by Sprenkle (1961) and reproduced by Smith Jr (1976) it follows that

$$\begin{aligned} c &= Se^{\rho T - rT} N(d_1) - Ke^{-rT} N(d_2)\,, \\ p &= Ke^{-rT} N(-d_2) - Se^{\rho T - rT} N(-d_1)\,, \\ \text{where} \quad d_1 &= \frac{\ln(S/K) + (\rho + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \\ \text{and} \quad d_2 &= \frac{\ln(S/K) + (\rho - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\,. \end{aligned} \tag{6.49}$$

where ρ is the expected average growth in the price of the foreign currency. From assumptions (i) and (iii) above, it follows that the forward rate F is an unbiased predictor of the spot rate at time T, hence

$$\rho T = \ln(F/S)\,. \tag{6.50}$$

When Equation (6.50) is substituted into Equation (6.49) it gives Equation (6.47).

Alternative Assumptions for Foreign Exchange Option Derivation

Garman and Kohlhagen (1983) developed alternative assumptions to the Black and Scholes (1973) model leading to a valuation formula for foreign exchange options. This valuation formula has a strong connection with the commodity-pricing model of Black (1976): when forward prices are given, and with the proportional dividend model of P. Samuelson and Merton (1969): when spot prices are given.

The Garman and Kohlhagen (1983) standard assumptions for a European option-pricing model are:

i. Geometric Brownian motion governs the currency spot price and the differential representation of spot price movements is $dS = \mu S dt + \sigma S dz$, where μ is the drift rate, σ the volatility and z is the standard Wiener process.

ii. Option prices are a function of only one stochastic variable, namely S.

iii. Markets are frictionless.

iv. Interest rates, both in the domestic and foreign markets, are constant.

The Garman and Kohlhagen (1983) approach to foreign exchange (FX) options relates the role of domestic and foreign interest rates by comparing the advantages of holding an FX option with those of holding its underlying currency. The risk-adjusted expected excess returns of securities must be identical in an arbitrage-free continuous-time economy, that is

$$\frac{\alpha_i - r_d}{\sigma_i^s} = \Upsilon, \quad \text{for all} \quad i\,. \tag{6.51}$$

where α_i is the expected return on a security, r_d the domestic riskless interest rate, σ_i^s the standard deviation of the security rate of return and where Υ does not depend on the security being considered. Applying this fact to the ownership of foreign currency, gives

$$\frac{(\mu + r_f) - r_d}{\sigma} = \Upsilon\,, \tag{6.52}$$

where the expected return from holding the foreign currency is the drift of the exchange rate in domestic units per foreign unit, μ, plus the riskless capital growth arising from holding the foreign currency in the form of an asset paying interest at the rate of r_f and σ is the standard deviation of the rate of return on holding the currency.

Letting $c(S,T)$ be the price of a European call option with time T left to maturity then

$$\frac{\alpha_c - r_d}{\sigma_c^s} = \Upsilon\,, \tag{6.53}$$

where α_c is the call option's expected rate of return and σ_c^s the options standard deviation. Applying Itô's lemma this becomes

$$\alpha_c c = \mu S \frac{\partial c}{\partial S} - \frac{\partial c}{\partial T} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial S^2} \tag{6.54}$$

and

$$\sigma_c^s S = \sigma S \frac{\partial c}{\partial S}\,. \tag{6.55}$$

Substituting Equations (6.54) and (6.55) into Equation (6.53) yields

$$\frac{\mu S \frac{\partial c}{\partial S} - \frac{\partial c}{\partial T} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial S^2} - r_d c}{\sigma \frac{\partial c}{\partial S}} = \Upsilon\,, \tag{6.56}$$

combining Equations (6.52) and (6.53) gives

$$\frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} - r_d c + (r_d S - r_f S)\frac{\partial c}{\partial S} = \frac{\partial c}{\partial T}\,. \tag{6.57}$$

Equation (6.57) is reminiscent of models proposed by P. Samuelson (1965) and P. Samuelson and Merton (1969) in which the dividend rate of a stock is presumed to be proportional to the level of the stock price. This interpretation can be applied for foreign currency options by regarding the foreign interest rate r_f, in terms of the domestic rate, as the dividend rate of the foreign currency.

The solutions to Equation (6.57) for European FX call options must obey the option boundary conditions $c(S,0) = max[S-K,0]$ and conversely the boundary conditions for European put options $p(S,0) = max[K-S,0]$ yielding the valuation formula

$$\begin{aligned} c &= Se^{-r_f T}N(d_1) - Ke^{-r_d T}N(d_2)\,, \\ p &= Ke^{-r_d T}N(-d_2) - Se^{-r_f T}N(-d_1)\,, \\ \text{where} \quad d_1 &= \frac{\ln(S/K) + (r_d - r_f + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \\ \text{and} \quad d_2 &= \frac{\ln(S/K) + (r_d - r_f - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\,. \end{aligned} \tag{6.58}$$

The valuation formulation for European FX options depend on both foreign and domestic interest rates. Note that the foreign interest rate r_f and the interest differential $r_d - r_f$ play distinct roles in the solution.

In the foreign currency markets, forward prices can involve either forward premiums or discounts because the forward value of a currency is related to the ratio of the prices of riskless bonds traded in each country. Accepting the Interest Rate Parity relationship, the forward price F of currency deliverable at the option maturity date T is

$$F = e^{(r_d - r_f)T} S \,. \tag{6.59}$$

Substituting Equation (6.59) into Equations (6.58) gives

$$\begin{aligned}
c(F,T) &= e^{-r_d T}\left[F N(d_1) - K N(d_2) \right] , \\
p(F,T) &= e^{-r_d T}\left[K N(-d_2) - F N(-d_1) \right] , \\
\text{where} \quad d_1 &= \frac{\ln(F/K) + (\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \\
\text{and} \quad d_2 &= \frac{\ln(F/K) - (\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} ,
\end{aligned} \tag{6.60}$$

where the option value depends only upon F and r_d and does not depend independently upon S and r_f. Given the current domestic interest rate, all option-relevant information concerning the foreign interest rate and the spot currency price is reflected in the forward price.

Garman and Kohlhagen (1983) concluded that the Black-Scholes option pricing model does not apply well to foreign exchange options since multiple interest rates are involved in ways differing from the Black-Scholes assumptions. Garman and Kohlhagen (1983) found that the appropriate valuation formulas for European FX options depend on both foreign and domestic interest rates.

Sørensen (1997) presented a modified version of the Garman and Kohlhagen (1983) formula where the equilibrium approach deviated from the no-arbitrage assumption by allowing domestic and foreign interest rates to be dynamically determined endogenously in the model. Within the Garman and Kohlhagen (1983) approach the drift parameter has no direct influence on the currency option price. The expected change in the exchange rate is represented in the option pricing formula through the domestic and foreign interest rates. Sørensen (1997) established a relationship between exchange rate dynamics and the dynamics of the domestic and foreign interest rates. If the foreign and domestic interest rates evolve according to Gaussian processes, it is possible to obtain closed-form pricing formula of the form derived by Garman and Kohlhagen (1983).

The Sørensen (1997) variant in the analysis is the characterisation of the volatilities for option pricing, which only depend on exchange rate parameters. The volatilities are expressed as a function of time and the parameters describing the variability of the log-exchange rate. This is possible due to the relationship between the exchange rate dynamics and the dynamics of the term structure of interest rates. From an equilibrium perspective, parameters in the drift term for the exchange rate may enter the Garman and Kohlhagen (1983) formula directly. Sørensen (1997) advocates that exchange rate parameters implicitly and explicitly affect currency option pricing.

6.2.6 Model Comparison and Risk Reversal Valuations

Pagès (1996) considered whether option pricing data could be useful in predicting exchange rate changes. Option prices reflect the market perceptions of the underlying asset's distribution, so they may reveal information about the future movement of exchange rates.

Among option-based forecast indicators in the foreign exchange market, two alternatives seem to be particularly relevant:

i. At-the-money volatility: the market's implicit volatility forecast for those options whose strike price is closest to being at the forward rate.

ii. The price of risk reversals: a derivative instruments constructed as a linear combination of out-of-the-money call and put options, written on the same currency and expiring at the same date.

This paper will focus on the risk reversal preference.

The payoff from risk reversals can be either negative or positive for large deviations of the exchange rate from the forward rate, depending on the direction of the move. For this reason, they are often interpreted as the market's best guess about the directional bias of future exchange rate movements.

Forecasts in the foreign exchange market are measured in terms of the variation from the forward rates, referred to as the forward bias. Pagès (1996) supports the view that information revealed by option prices helps improve forecasts of future spot rates. Pagès (1996) points to a correlation between risk reversals and the forward bias: when the price of risk reversals goes up, the leading currency's forward rate tends to increase with respect to future realisations of the spot rate. If there is a downward forward bias, implying that on average the forward rate is below the future spot rates, the bias will be reduced. Conversely, if the bias is upwards, indicating an overestimation of realised future spot rates, the bias will be increased.

Risk reversals capture the skewness of the exchange rate distribution. Skewness in option pricing can cause a risk reversal that theoretically should have no cost to have a positive or negative cost. Malz (1996) asserts that the skewness is an indication of market sentiment regarding the likelihood of a currency appreciating or depreciating. The risk reversal cost can be regarded in one of two ways:

i. A positive or negative net cost.

ii. If the purchaser of the risk reversal is unwilling to incur upfront costs, then the strike prices of the risk reversal may need to be adjusted.

This analysis will assume the first approach.

Traders often portray the skewness in volatility terms, and refer to the risk reversal cost as the difference in volatility. The skewness of a volatility smile in equally out-of-the money call and put options may be caused by differences in their respected volatilities: Malz (1997).

Exchange rate shocks seem to be asymmetric, in that their magnitude tends to vary according to whether the spot rate appreciates or depreciates. Classical option pricing theory requires that exchange rates follow a geometric Brownian motion with constant second moments. Market-makers have known that this key assumption is flawed and use standard implied volatilities only as a convenient way to price options. Empirical investigations concerning nominal returns: Westerfield (1977), found that nominal exchange returns violate the normality assumption with respect to:

i. The distribution having a time-varying variance, with contiguous periods of high and low volatility.

ii. Having fat tails, implying that for a given variance there is a higher probability of large deviations from the mean.

iii. It is skewed, in that an appreciation and a depreciation of a given size are not equally likely.

As a result, alternative models have been developed to generalise the Black-Scholes formula by allowing volatility to change randomly: J. Hull and White (1987), Scott (1987) and Wiggins (1987). Melino and Turnbull (1991) found that these models can explain the price of currency options, although they tended to overestimate volatility.

The Pagès (1996) empirical results asserted that the forward biases were positively correlated with risk reversals. Risk reversals capture the directional biases in the exchange rate with respect to the risk-neutral probability and conditional on large deviations from the forward rate. Risk reversals do capture the skewness of the distribution. An interpretation is that the volatility is stochastic and its time-varying correlation with the exchange rate induces skewness in the distribution. This is precisely what makes risk reversals valuable. Hence, the price changes of risk reversals reflect the time-varying correlation between exchange rate risk and volatility risk.

Dunis and Lequeux (2001) investigated whether there was any informational value that can be derived from the price of risk reversals: the volatility amount by which a 25-delta call is more or less expensive than a 25-delta put, and use this information to assess the future evolution of exchange rates. Risk reversals are a measure of the skewness of the distribution and directly observable. The possibility of positively correlated directional information embedded in risk reversals values derives from the very high level of contemporaneous correlation rather than from lagged information that may exist between the risk reversals and the underlying time series.

Risk reversals are directional option strategies. The strategy involves the simultaneous sale of an out-of-the-money put and purchase of an out-of-the-money call, of the same maturity.

These strategies are usually built around options that have a 1-month or 3-month time value and that have out-of-the-money strikes corresponding to 25% delta. Option delta values are an indication of the degree the option is in-the-money or out-of-the-money. The delta rises as options are increasingly in-the-money and reduce as the options move progressively out-of-the money. At-the-money options have a value of 50-delta, suggesting a 50% likelihood of ending up either in-the-money or out-of-the-money at maturity. Hence a 25-delta option is out-of-the-money with only a 25% chance of ending in-the-money at maturity. Delta is the change in the value of an option for an infinitesimal change in the exchange rate. A standard out-of-the-money currency option offered in the interbank market is a 25-delta option, which means that the price of the option would change 0.25% for a 1% change in the exchange rate. Thus a 25-delta option is an indication of the degree an option is out-of-the-money.

Dunis and Lequeux (2001) investigated the economic implications of using risk reversals as a source of information to trade the spot foreign exchange market. Dunis and Lequeux (2001) examined the two most common assumptions made by market practitioners that risk reversals are either:

i. Directional trend indicators.

ii. Early trend reversal indicators.

Technical trading strategies, chartism, enable traders to make financial decisions without relying on fundamental analysis. Chartists suppose that the price contains all the information pertaining to the market. Therefore the price incorporates the fundamental views by default. The chartist hypothesises that all available information and economic data are taken into account in the exchange rate and hence the pricing of the risk reversal.

To evaluate the economic viability of these two assumptions, Dunis and Lequeux (2001) devised trading strategies using risk reversal levels and compared the returns obtained to those generated by using a simple 21-day moving average trading strategy on the spot exchange rate itself.

The alternative FX option pricing model will be evaluated by comparing the risk reversals derived from this model. Utilising the Dunis and Lequeux (2001) trading strategies to forecast the spot market, the modified risk reversals forecasting performance will be compared to the BSM market derived risk reversals. The premise being that an improved model should be used more profitably within a directional forecasting context: the better model is assumed to afford improved forecasting abilities.

6.3 Conclusions

A common feature of the established FX option pricing models in the market today is the acceptance of geometric Brownian motion as the diffusion process on which they are based. The Black and Scholes (1973) model is the defining element in option pricing theory. However a number of the BSM assumptions have been challenged in light of the discrepancies between some of these assumptions and what is seen in the market. Rather than enhance or extend the Black and Scholes (1973) model to allow for these discrepancies the diffusion process on which the model is based has been examined.

This chapter was concerned with the main area underpinning the discrepancy between the market price and the BSM option price, that of the stochastic price process governing the behaviour of the underlying asset. To identify the foreign exchange price processes, the data will be transformed to comply with the assumption that the price process follows a geometric Brownian motion of the form shown in Equation (B.16) on page 156. The Kon (1984) shift parameter will be applied to the system of normal distributions to transform the data, generating a modified geometric Brownian motion process. The system of normal distributions can then be used to address the discrepancy between the theoretical model and the actual price distributions. The modified price process will then be applied to the Black and Scholes (1973) method to arrive at a closed form option pricing model that can be utilised by traders.

Due to insufficient option pricing data, the modified model will be evaluated by comparing the risk reversal derived from it to that of the market derived risk reversals using the trading strategies outlined by Dunis and Lequeux (2001). The performance criterion is founded on the premise that the better model will more accurately reflect the information contained in the market in forecasting the foreign exchange directional changes.

Chapter 7

A New Approach to Pricing FX Options

There are many derivations of the Black–Scholes equation for the price of a European call option. The methods used for proving the Black–Scholes formula fall into two categories:

i. The bond replication method: Black and Scholes (1973).

ii. The call replication method: R. C. Merton (1976).

These two methods are not equivalent. The bond replication method contains greater restrictions on the call price: notably that the call price is reasonably bounded and a function of the spot price and time, whereas the call replication method only requires the continuity of the call option price.

Rosu and Stroock (2003) considered the differences between the two methods. Accepting that the Black and Scholes (1973) environment contains a stock S, a bond B, and a European call option C then the bond replication method requires that the hedged portfolio Π involved in the arbitrage argument is both riskless and self-financing. Let a portfolio formed with the stock and call be $\Pi = aS - bC$ which replicates a riskless and self-financing bond. The self-financing condition implies $d\Pi = a\,dS - b\,dC$ which together with the riskless condition gives $\Pi = b(SC - C)$. An arbitrage argument shows that Π must earn interest at the riskless rate r, hence $d\Pi = r\Pi\,dt$. Applying Itô's process, which is a generalised Wiener stochastic process where the parameters are functions of the stock S and time t, the call option price C must satisfy the Black-Scholes partial differential equation.

Conversely the call replication method proceeds by attempting to replicate the call option price by forming a portfolio Θ with the stock and the bond such that $\Theta = \alpha S + \beta B$ has the same payoff as C at maturity. If Θ is self-financing then the arbitrage argument implies that the call option price C should equal the portfolio Θ at any given time. Applying Itô's process the replicating conditions determines Θ at maturity as the solution to the Black-Scholes equation.

The Black and Scholes (1973) bond replication and the R. C. Merton (1976) call replication methods for proving the Black-Scholes formula assume the stock price S follows a normally distributed geometric Brownian motion. Employing the Kon (1984) model, the stock price behaviour will be amended to reflect the discrepancy between the theoretical and observed price distributions. The Kon (1984) discrete mixture of normal distributions, whose drift parameter μ and volatility σ are time dependent and shift along a finite number of values determined from real data, will be used to identify the foreign exchange price process. This price process will be the basis of an FX option pricing model based on the Black-Scholes methodology which will be proved using the original bond replication method.

7.1 The Modified Distribution of Foreign Exchange

7.1.1 The Discrete Mixture of Normal Distributions Modification to GBM

Kon (1984) argued that the distribution of stock returns may be normal with parameter shifts among a finite set of values. The parameters shifts explain the observed skewness: due to differences in the mean, and the kurtosis: due to differences in the variance. Noting that stationarity tests on the parameter estimates revealed significant differences in the mean and variance of the daily return distribution, Kon (1984) proposed the validity of the discrete mixture of normal distributions process as a statistical model for stock returns.

To identify the parameters of the normal distribution Kon (1984) assumed each return observation is drawn from one of N sets of parameter values. The *EViews 11* stability diagnostic: multiple breakpoint test estimation procedure was employed to identify the N normal distributions. Tests for parameter instability and structural change are an important part of applied econometric work. The stability test examines whether the parameters of the discrete normal model are stable across sub-samples of the data. In some cases there may be obvious points at which to determine a break in the data: a financial shock, a switch from fixed to floating exchange rates, *etc.* Where there is no obvious break point econometric methods are applied.

Where Chow (1960) developed a test for regime change at known dates, Quandt (1960) and Andrews (2003) modified the Chow (1960) framework deriving the limiting distribution for the largest value over all possible breakpoint dates. Bai (1997) and Bai and Perron (1998) further provided a theoretical and computational result that extended the Quandt-Andrews framework by allowing multiple unknown breakpoints. To use the tools for testing multiple breakpoints the equation is initially estimated by the method of least squares. From the estimated equation the global information criterion uses the information computed from the global optimisers to determine the number of breaks. The Yao (1988) information criterion shows that the number of breaks that minimises the Schwarz criterion is a consistent estimator of the true number of breaks. The output gives the global optimiser: estimated breakpoint dates for each number of breaks, the number of estimated coefficients and the logarithmic likelihood along with the Schwarz (SIC) and the Liu, Wu and Zidex (LWZ) information criterion and the sum of squared residuals.

For each of the N-distributions the parameter estimates are differenced *see* Table 5.5 on page 80. The shift in the mean and variance parameters explain the change in the investment and financial decision variables. This change results in an adjustment to the expected return and the standard deviation of the return distribution observed in the markets.

Under the assumption that geometric Brownian motion is the ideal model for the stock price returns, then allowing for the shift in the parameters suggests that the modified geometric Brownian motion, incorporating the parameter shifts, is a better representation of the return distribution, given by

$$dS = (\mu + \mu_{shift})\, Sdt + \sqrt{(\sigma^2 + \sigma^2_{shift})}\, Sdz\,, \tag{7.1}$$

where μ is the entire sample period mean and σ^2 the corresponding variance and μ_{shift} and σ^2_{shift} is the shift in the mean and variance corresponding to the N^{th} distribution.

7.1.2 The Modified GBM Derivation of the BSM Equation

A fundamental assumption of the Black and Scholes (1973) formula is that the stochastic process for the stock price followed a geometric Brownian motion. A model for FX spot rates must allow for the stochastic price behaviour and strict positivity. These are the same requirements applied to the BSM stock model. Therefore these requirements follow the Black and Scholes (1973) and the associated work of Garman and Kohlhagen (1983) in describing the spot rate by a geometric Brownian motion. However the assumption here is that the price process is the Kon (1984) parameter shifted modified geometric Brownian motion given by

$$\frac{dS}{S} = (\mu + \mu_{shift})\, dt + \sqrt{(\sigma^2 + \sigma^2_{shift})}\, dz\,, \tag{7.2}$$

where S is the FX spot rate and dS/S follows a modified geometric Brownian motion with expected rate of return $(\mu + \mu_{shift})$ and variance $(\sigma^2 + \sigma^2_{shift})$.

A fundamental principle of the Black and Scholes (1973) formula is the creation of a riskless portfolio consisting of a certain proportion of shares and options satisfying the boundary conditions. The situation in foreign exchange is more complicated in that the FX spot rate S is not a store of wealth and cannot be regarded as a tradeable stock. Instead exchange rates should be regarded as a stochastic conversion rate relating two numeraire currencies, each with a money market account in either currency. In this case the contingent claim $f(S,t)$ derives its value from the performance of an FX rate S where the tradeable asset is not the FX rate S but rather the foreign bond B_f valued in units of the domestic currency: SB_f. The expected rate of return now becomes the difference between the risk-free rates of the respective currencies.

Suppose that f is the price of a contingent claim on FX where f is a function of the FX rate S and t. Using this assumption and letting $f(S,t)$ denote the price of the option derived from the spot exchange rate S at time t; applying Itô's giving

$$df = \frac{\partial f}{\partial t}\, dt + \frac{\partial f}{\partial S}\, dS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\, dS^2\,, \tag{7.3}$$

where $dS^2 = (\sigma^2 + \sigma^2_{shift})S^2\,dt$. Thus

$$df = \left[\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(\sigma^2 + \sigma^2_{shift})S^2\right] dt + \frac{\partial f}{\partial S}\, dS\,. \tag{7.4}$$

The term inside the brackets is deterministic, whereas the term in front of the dS is the only stochastic term. Black and Scholes (1973) removed the stochastic term by constructing a portfolio Π which is short one unit of the contingent claim f and long $\partial f/\partial S$ units of the underlying asset

$$\Pi = -f + \frac{\partial f}{\partial S}\, S\,. \tag{7.5}$$

The construction of the delta-hedged portfolio for foreign exchange is somewhat different.

Proposition 1: There is no natural numeraire currency for foreign exchange rates; therefore a delta-hedged portfolio must incorporate a foreign currency bond as the arbiter of value. Thus the exchange rate is now determined by the value of the bond and the spot FX rate. The fundamental equation now becomes,

$$\left[\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\left(\sigma^2 + \sigma^2_{shift}\right) S^2 + \frac{\partial f}{\partial S}\left(r_d - r_f\right) S - r_d f\right] = 0\,. \tag{7.6}$$

Proof: The construction of the hedged portfolio Π is obtained by going short one unit of the contingent claim f and long Δ units of the foreign bond valued in units of the domestic currency

$$\Pi = -f + \Delta\, SB_f\,. \tag{7.7}$$

The change $d\Pi$ in the value of the portfolio over the interval dt is given by

$$d\Pi = -df + \Delta\, d(SB_f)\,, \tag{7.8}$$

expanding gives

$$d\Pi = -df + \Delta B_f\, dS + \Delta S\, dB_f\,. \tag{7.9}$$

Given the return on the foreign bond is the risk-free foreign interest rate over time: $dB_f = r_f B_f\, dt$ then

$$d\Pi = -df + \Delta B_f\, dS + \Delta S\, r_f B_f\, dt\,. \tag{7.10}$$

Applying the modified geometric Brownian motion dS

$$d\Pi = -df + \Delta B_f\, \left[(r_d - r_f)\, S\, dt + \sqrt{\sigma^2 + \sigma^2_{shift}}\, S\, dz\right] + \Delta S\, r_f B_f\, dt\,, \tag{7.11}$$

expanding

$$d\Pi = -df + \Delta B_f\,(r_d - r_f)\, S\, dt + \Delta B_f \sqrt{\sigma^2 + \sigma^2_{shift}}\, S\, dz + \Delta S\, r_f B_f\, dt\,, \tag{7.12}$$

collecting the terms

$$d\Pi = -df + \left[\Delta B_f\,(r_d - r_f)\, S + \Delta S\, r_f B_f\right] dt + \Delta B_f \sqrt{\sigma^2 + \sigma^2_{shift}}\, S\, dz\,, \tag{7.13}$$

expanding

$$d\Pi = -df + [\Delta B_f r_d\, S - \Delta B_f r_f\, S + \Delta r_f B_f\, S]\, dt + \Delta B_f \sqrt{\sigma^2 + \sigma^2_{shift}}\, S\, dz\,, \tag{7.14}$$

and simplifying

$$d\Pi = -df + \Delta\, B_f r_d\, S\, dt + \Delta B_f \sqrt{\sigma^2 + \sigma^2_{shift}}\, S\, dz\,, \tag{7.15}$$

collecting the terms

$$d\Pi = -df + \Delta\, B_f \left[r_d\, S\, dt + \sqrt{\sigma^2 + \sigma^2_{shift}}\, S\, dz \right]\,, \tag{7.16}$$

expanding Itô's lemma to the diffusion process df

$$d\Pi = \left[-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\left(\sigma^2 + \sigma^2_{shift}\right) S^2 \right] dt - \frac{\partial f}{\partial S}\, dS + \Delta B_f \left[r_d\, S\, dt + \sqrt{\sigma^2 + \sigma^2_{shift}}\, S\, dz \right]\,, \tag{7.17}$$

expanding and collecting the terms

$$d\Pi = \left[-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\left(\sigma^2 + \sigma^2_{shift}\right) S^2 + \Delta B_f r_d\, S \right] dt - \frac{\partial f}{\partial S}\, dS + \Delta B_f \sqrt{\sigma^2 + \sigma^2_{shift}}\, S\, dz\,, \tag{7.18}$$

applying the modified geometric Brownian motion dS

$$\begin{aligned} d\Pi = &\left[-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\left(\sigma^2 + \sigma^2_{shift}\right) S^2 + \Delta B_f r_d\, S \right] dt - \frac{\partial f}{\partial S} \left[(r_d - r_f) S\, dt + \sqrt{\sigma^2 + \sigma^2_{shift}}\, S\, dz \right] \\ &+ \Delta B_f \sqrt{\sigma^2 + \sigma^2_{shift}}\, S\, dz\,, \end{aligned} \tag{7.19}$$

expanding

$$\begin{aligned} d\Pi = &\left[-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\left(\sigma^2 + \sigma^2_{shift}\right)\, S^2 + \Delta B_f\, r_d\, S \right] dt - \frac{\partial f}{\partial S}(r_d - r_f)\, S\, dt \\ &- \frac{\partial f}{\partial S}\sqrt{\sigma^2 + \sigma^2_{shift}}\, S\, dz + \Delta B_f \sqrt{\sigma^2 + \sigma^2_{shift}}\, S\, dz\,, \end{aligned} \tag{7.20}$$

collecting the dt and dz terms

$$\begin{aligned} d\Pi = &\left[-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\left(\sigma^2 + \sigma^2_{shift}\right)\, S^2 - \frac{\partial f}{\partial S}\left(r_d - r_f\right)\, S + \Delta B_f\, r_d\, S \right] dt \\ &- \left[\frac{\partial f}{\partial S} - \Delta B_f \right] \sqrt{\sigma^2 + \sigma^2_{shift}}\, S\, dz\,. \end{aligned} \tag{7.21}$$

To cancel the dz term, Δ must satisfy $\frac{\partial f}{\partial S} = \Delta B_f$, that is

$$\Delta = \frac{1}{B_f} \cdot \frac{\partial f}{\partial S}\,. \tag{7.22}$$

Substituting Equation (7.22) into (7.21) gives

$$\begin{aligned} d\Pi = &\left[-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\left(\sigma^2 + \sigma^2_{shift}\right)\, S^2 - \frac{\partial f}{\partial S}\left(r_d - r_f\right)\, S + \frac{1}{B_f}\frac{\partial f}{\partial S} B_f r_d\, S \right] dt \\ &- \left[\frac{\partial f}{\partial S} - \frac{1}{B_f}\frac{\partial f}{\partial S} B_f \right] \sqrt{\sigma^2 + \sigma^2_{shift}}\, S\, dz\,, \end{aligned} \tag{7.23}$$

simplifying

$$d\Pi = \left[-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2} \left(\sigma^2 + \sigma^2_{shift}\right) S^2 - \frac{\partial f}{\partial S} \left(r_d - r_f\right) S + \frac{\partial f}{\partial S} r_d \, S \right] dt \, . \tag{7.24}$$

Applying domestic risk neutrality the portfolio return is given as

$$d\Pi = r_d \, \Pi dt \;\text{ where }\; \Pi = -f + \frac{\partial f}{\partial S} S \, . \tag{7.25}$$

Therefore giving

$$\left[-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2} \left(\sigma^2 + \sigma^2_{shift}\right) S^2 - \frac{\partial f}{\partial S} \left(r_d - r_f\right) S + \frac{\partial f}{\partial S} r_d \, S \right] dt = r_d \left[-f + \frac{\partial f}{\partial S} S \right] dt \, , \tag{7.26}$$

expanding and cancelling dt gives

$$\left[-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2} \left(\sigma^2 + \sigma^2_{shift}\right) S^2 - \frac{\partial f}{\partial S} \left(r_d - r_f\right) S + \frac{\partial f}{\partial S} r_d \, S \right] = \left[-r_d f + \frac{\partial f}{\partial S} r_d S \right] , \tag{7.27}$$

simplifying

$$\left[\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2} \left(\sigma^2 + \sigma^2_{shift}\right) S^2 + \frac{\partial f}{\partial S} \left(r_d - r_f\right) S - \frac{\partial f}{\partial S} r_d \, S + \frac{\partial f}{\partial S} r_d \, S - r_d f \right] = 0 \, , \tag{7.28}$$

which reduces to

$$\left[\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2} \left(\sigma^2 + \sigma^2_{shift}\right) S^2 + \frac{\partial f}{\partial S} \left(r_d - r_f\right) S - r_d f \right] = 0 \, . \tag{7.29}$$

Note that the foreign interest rate r_f appears in the convection term but not in the forcing term and the absence of any μ term. The derivative is obtained when the partial differential equation (PDE) is solved with boundary condition $f = max(S - K, 0)$ for the call option and $f = max(K - S, 0)$ for the put.

The Black-Scholes form of the PDE describes how the value of a derivative contract, at a continuum of future prices, is diffused backwards in time to the present. The present value of a derivative in units of the domestic currency is identified as the discounted expectation under the domestic risk-free measure

$$f_0 = e^{-r_d T} E_d \left[f_T \right] \, , \tag{7.30}$$

where f_0 is the present value of the derivative at time $t = 0$ and f_T the terminal value discounted at the risk-neutral domestic measure r_d and E_d is the expectation with respect to the domestic risk neutral rate. The risk-neutral measure, assumed to take a particular value of drift rate μ in the price of a risky asset, so that an investor's expectations of the returns of the two assets available to them are identical. But what are these two assets in foreign exchange? Foreign exchange (FX) has the choice of two bonds: the domestic bond B_d and the foreign bond B_f, where the FX spot rate must be used to convert one into the numeraire currency. Thus this gives us the choice of two risk-neutral measures in foreign exchange.

7.1.3 The Valuation of European FX Options

In obtaining the modified Black-Scholes partial differential equation (PDE), *see* Equation (7.29), the real-world drift term $(\mu + \mu_{shift})$ does not appear. The equation does not involve any variables that are affected by the risk preferences of the investors. Therefore it can be assumed that all rational investors arrive at the same price for the derivative irrespective of the value of the expected drift rate. The risk-neutrality is an important assumption for the analysis of derivatives. The PDE does not contain the risk preferences of investors, the variables that do appear are:

i. Current spot price S,

ii. Time to maturity T.

iii. Variance $(\sigma^2 + \sigma^2_{shift})$.

iv. Risk-free domestic rate r_d and foreign rate r_f.

Proposition 2: Given that there is no natural numeraire in FX and the investors risk preferences do not appear, then the FX option pricing formula must incorporate the FX risk-neutrality comprised by the choice of two bonds: the domestic bond B_d and the foreign bond B_f, linked by the FX spot rate.

Proof: From Clark (2011) consider a European call option with payout $f_T = max(S_T - K, 0) = (S_T - K)^+$ at time T. The present value of the option price would be

$$f_0 = e^{-r_d T} E_d\left[(S_T - K)^+\right] , \tag{7.31}$$

given the indicator function: indicates membership of an element in a subset, $1_{(S \geq K)}$ of an event $(S \geq K)$ occurring then

$$f_0 = e^{-r_d T} E_d\left[(S_T - K) 1_{(S \geq K)}\right] , \tag{7.32}$$

expanding

$$f_0 = e^{-r_d T} E_d\left[S_T 1_{(S \geq K)} - K 1_{(S \geq K)}\right] , \tag{7.33}$$

taking discounted expectations

$$f_0 = e^{-r_d T} E_d\left[S_T 1_{(S \geq K)}\right] - K e^{-r_d T} E_d[1_{(S \geq K)}] , \tag{7.34}$$

given that the expectation of the indicative function is equal to the domestic risk-neutral probabilities $E_d\left[1_{(S \geq K)}\right] = P_d(S_T \geq K)$ and simplifying

$$f_0 = e^{-r_d T} E_d\left[S_T 1_{(S \geq K)}\right] - K e^{-r_d T} P_d\left[S_T \geq K\right] . \tag{7.35}$$

Taking each part of Equation (7.35) in tern.

Computation of the expectation term $E_d\left[S_T 1_{(S \geq K)}\right]$, taking the properties of the logarithmic price S_T from Equation (4.2) on page 41 and the variance shift parameter gives

$$E_d\left[S_T 1_{(S \geq K)}\right] = E_d\left[S_0 \exp\left(\left(r_d - r_f - \frac{1}{2}\left(\sigma^2 + \sigma^2_{shift}\right)\right) T + \sqrt{\sigma^2 + \sigma^2_{shift}}\, z_{T(d)}\right) 1_{(S \geq K)}\right] . \tag{7.36}$$

Letting P_d represent the domestic risk-neutral measure and P_f the foreign risk-neutral measure, the derivative relating one risk-neutral measure to the other is given as

$$\frac{dP_f}{dP_d} = \exp\left(\sqrt{\sigma^2 + \sigma^2_{shift}}\, z_{T(d)} - \frac{1}{2}(\sigma^2 + \sigma^2_{shift})\, T\right) . \tag{7.37}$$

Isolating the risk-neutral relationship

$$E_d\left[S_T 1_{(S\geq K)}\right] = S_0 e^{(r_d - r_f)T} E_d\left[\exp\left(\sqrt{\sigma^2 + \sigma^2_{shift}}\, z_{T(d)} - \frac{1}{2}\left(\sigma^2 + \sigma^2_{shift}\right)\, T\right) 1_{(S\geq K)}\right] , \tag{7.38}$$

simplifying

$$E_d\left[S_T 1_{(S\geq K)}\right] = S_0 e^{(r_d - r_f)T} E_d\left[\frac{dP_f}{dP_d}\, 1_{(S\geq K)}\right] , \tag{7.39}$$

relating the foreign expectations

$$E_d\left[S_T 1_{(S\geq K)}\right] = S_0 e^{(r_d - r_f)T} E_f\left[1_{(S\geq K)}\right] , \tag{7.40}$$

given $E\left[1_{(S\geq K)}\right] = P\left[S \geq K\right]$, yields

$$E_d\left[S_T 1_{(S\geq K)}\right] = S_0 e^{(r_d - r_f)T} P_f\left[S \geq K\right] . \tag{7.41}$$

Substituting Equation (7.41) into (7.35) gives

$$f_0 = S_0 e^{-r_f T} P_f\left[S_T \geq K\right] - K e^{-r_d T} P_d\left[S_T \geq K\right] , \tag{7.42}$$

where the two risk-neutral probabilities, P_f and P_d, need to be calculated so that $[S_T \geq K]$. Noting that the FX risk-neutrality comprises a choice of two bonds: the domestic bond B_d and the foreign bond B_f, linked by the FX spot rate, the probabilities are computed from the risk-neutral measure $P_d\left[S_T \geq K\right]$ and $P_f\left[S_T \geq K\right]$.

The domestic risk-neutral measure P_d incorporating the shift parameter is given as

$$S_T = S_0 \exp\left(\sqrt{\sigma^2 + \sigma^2_{shift}}\, z_{T(d)} + \left(r_d - r_f - \frac{1}{2}\left(\sigma^2 + \sigma^2_{shift}\right)\right) T\right) , \tag{7.43}$$

substituting S_T into $P_d\left[S_T \geq K\right]$ gives

$$P_d\left[S_0 \exp\left(\sqrt{\sigma^2 + \sigma^2_{shift}}\, z_{T(d)} + \left(r_d - r_f - \frac{1}{2}\left(\sigma^2 + \sigma^2_{shift}\right)\right) T\right) \geq K\right], \tag{7.44}$$

divide by S_0

$$P_d\left[\exp\left(\sqrt{\sigma^2 + \sigma^2_{shift}}\, z_{T(d)} + \left(r_d - r_f - \frac{1}{2}\left(\sigma^2 + \sigma^2_{shift}\right)\right) T\right) \geq \frac{K}{S_0}\right] , \tag{7.45}$$

taking logarithms

$$P_d\left[\sqrt{\sigma^2 + \sigma^2_{shift}}\, z_{T(d)} + \left(r_d - r_f - \frac{1}{2}\left(\sigma^2 + \sigma^2_{shift}\right)\right) T \geq \ln\left(\frac{K}{S_0}\right)\right], \tag{7.46}$$

$$P_d\left[\sqrt{\sigma^2+\sigma^2_{shift}}\,z_{T(d)} \geq \ln\left(\frac{K}{S_0}\right)-\left(r_d-r_f-\frac{1}{2}\left(\sigma^2+\sigma^2_{shift}\right)\right)T\right], \tag{7.47}$$

$$P_d\left[-\sqrt{\sigma^2+\sigma^2_{shift}}\,z_{T(d)} \leq \ln\left(\frac{S_0}{K}\right)+\left(r_d-r_f-\frac{1}{2}\left(\sigma^2+\sigma^2_{shift}\right)\right)T\right], \tag{7.48}$$

given $z=\epsilon\sqrt{T}$ where $\epsilon \sim (0,1)$ is a standardised normal random variable, noting that $\epsilon\sqrt{T}$ and $-\epsilon\sqrt{T}$ have the same distribution by symmetry

$$P_d\left[\sqrt{\sigma^2+\sigma^2_{shift}}\,\epsilon\sqrt{T} \leq \ln\left(\frac{S_0}{K}\right)+\left(r_d-r_f-\frac{1}{2}\left(\sigma^2+\sigma^2_{shift}\right)\right)T\right], \tag{7.49}$$

$$P_d\left[\epsilon \leq \frac{\ln\left(\frac{S_0}{K}\right)+\left(r_d-r_f-\frac{1}{2}\left(\sigma^2+\sigma^2_{shift}\right)\right)T}{\sqrt{\sigma^2+\sigma^2_{shift}}\,\sqrt{T}}\right]. \tag{7.50}$$

Therefore domestic risk-neutral probability d_2 is

$$d_2=\frac{\ln\left(\frac{S_0}{K}\right)+\left(r_d-r_f-\frac{1}{2}\left(\sigma^2+\sigma^2_{shift}\right)\right)T}{\sqrt{\sigma^2+\sigma^2_{shift}}\,\sqrt{T}}. \tag{7.51}$$

And for the foreign risk-neutral measure P_f incorporating the shift parameter,

$$S_T=S_0\exp\left(\sqrt{\sigma^2+\sigma^2_{shift}}\,z_{T(f)}+\left(r_d-r_f+\frac{1}{2}\left(\sigma^2+\sigma^2_{shift}\right)\right)T\right), \tag{7.52}$$

by symmetry the foreign risk-neutral probability d_1 is

$$d_1=\frac{\ln\left(\frac{S_0}{K}\right)+\left(r_d-r_f+\frac{1}{2}\left(\sigma^2+\sigma^2_{shift}\right)\right)T}{\sqrt{\sigma^2+\sigma^2_{shift}}\,\sqrt{T}}. \tag{7.53}$$

Given that $P_d\left[S_T \geq K\right]=N(d_1)$ and $P_f\left[S_T \geq K\right]=N(d_2)$ where $N(d_1)$ and $N(d_2)$ are the cumulative distribution functions, substituting into Equation (7.42) gives the European call option valuation

$$f_{c0}=S_0e^{-r_fT}N(d_1)-Ke^{-r_dT}N(d_2). \tag{7.54}$$

Conversely the European put option valuation is given by

$$f_{p0}=Ke^{-r_dT}N(-d_2)-S_0e^{-r_fT}N(-d_1). \tag{7.55}$$

This gives the modified FX option pricing model as

$$c = Se^{-r_f T} N(d_1) - Ke^{-r_d T} N(d_2)\,,$$

$$p = Ke^{-r_d T} N(-d_2) - Se^{-r_f T} N(-d_1)\,,$$

$$\text{where} \quad d_1 = \frac{\ln(S/K) + (r_d - r_f + \frac{1}{2}(\sigma^2 + \sigma^2_{shift}))T}{\sqrt{\sigma^2 + \sigma^2_{shift}}\sqrt{T}} \tag{7.56}$$

$$\text{and} \quad d_2 = \frac{\ln(S/K) + (r_d - r_f - \frac{1}{2}(\sigma^2 + \sigma^2_{shift}))T}{\sqrt{\sigma^2 + \sigma^2_{shift}}\sqrt{T}}\,.$$

This model will be evaluated by comparing the performance of the risk reversals derived from this model using the Dunis and Lequeux (2001) trading strategies to forecast the spot market against the BSM market derived risk reversals. The premise being that an improved model should be used more profitably within a directional forecasting context; that is the better model should afford improved forecasting abilities.

7.2 Data Set and Methodology

Historic FX option pricing data is not readily available from the Eikon system. To compare the performance of the new Modified FX option pricing model against the established Garman and Kohlhagen (1983) modified BSM model requires the use of risk reversals. Using the daily FX spot price data for the six currency pairs: EUR/USD, EUR/GBP, GBP/USD, USD/JPY, EUR/JPY and GBP/JPY, the corresponding call and put option price will be calculated using the modified FX option pricing model. These option prices will then be used to determine alternative synthetic risk reversal prices. The forecasting performance of the synthetic risk reversals will be compared to the historic BSM risk reversals from the Eikon system to determine which more accurately forecast the movement and direction of exchange rates.

The Refinitiv Eikon trading system, formerly Thomson-Reuters, provided the data for this study. The data set covers the 10 year period from July 22nd, 2010, to February 19th, 2020 for EUR/GBP, EUR/USD, GBP/USD, USD/JPY and EUR/JPY consisting of 2,500 observations and the 7 years from March 12th, 2013, to February 19th, 2020 for GBP/JPY consisting of 1,813 observations. The data comprises the top four currencies by percentage share of average daily turnover as per the April 2019 BIS survey, *see* Table 3.3 on page 31. The related currency pairs consists of EUR/USD, EUR/GBP, GBP/USD, USD/JPY, EUR/JPY and GBP/JPY, *see* Table 3.2 on page 31. The exchange rate and risk reversal price history comprises the best bid, mid and ask quotes and time-stamped to the daily closing price. No information as to the transaction size or trading parties is given. The corresponding LIBOR rates for each of the four currencies covering the sample period were downloaded from http://iborate.com.

Volatility is a measure of the uncertainty about the returns and defined as the standard deviation of the return in one year when the return is expressed using continuous compounding. This cannot be observed and has proved to be the most problematic element of the BSM model which assumes volatility is constant. Market practitioners attempt to address this limitation by inverting the model to calculate the implied volatility. These are volatilities implied by option prices observed in the market and used to price risk reversals.

A risk reversal is a commonly used term in the FX markets and refers to:

i. An option strategy combining the simultaneous purchase of out-of-the-money calls (puts) with the sale of out-of-the-money puts (calls). The option will have the same expiration date and similar deltas.

ii. A market view on both the underlying currency and implied volatility.

The risk reversal is founded on the limitations of the BSM option pricing model. The BSM model assumes a constant variance and that the price returns are normally distributed. Given that prices are in fact skewed and leptokurtic, the underlying currency will trade at extreme prices from the current spot market more frequently than a normal distribution would suggest, referred to as fat-tails.

If more price action occurs at these extreme levels than suggested by the BSM model, market traders mark the implied volatility higher to account for the increased probability of the underlying trading in the fat-tails region. This results in the implied volatility being higher for out-of-the-money and in-the-money options than for at-the-money options. If there is no bias in the market expectations of the underlying price the volatility would be symmetrical around the at-the-money volatility, commonly referred to as the volatility smile. When bias is introduced due to demand and supply considerations along with market expectations, this pushes up the volatility for in-the-money calls and out-of-the-money puts relative to in-the-money puts and out-of-the-money calls. This is referred to as the volatility skew.

Risk reversals express the difference in implied volatility of a 25-delta call and 25-delta put option

$$RR_{25} = \sigma_{call,25} - \sigma_{put,25}\,. \tag{7.57}$$

If the currency is expected to appreciate, calls would be favoured over puts and the purchaser of the call would pay a higher volatility relative to puts.

Consider a different view of volatility to current market practice. Whereas risk reversals are priced by differencing the implied volatility, assume that the volatility is the annualised standard deviation for the entire sample period, adjusted for the parameter shift due to the stability of the N normal distributions of the Kon (1984) model. Therefore the pricing of the risk reversal derived from my modified FX option pricing formula is formed by differencing the option price rather than differencing the implied volatilities. The delta of an option measures the amplitude of the change of its price as a function of the change in the price of its underlying asset. 25-delta risk reversals are priced by differencing out-of-the-money 25-delta call and put options where 50-delta represents an at-the-money strike price. The strike price K for the modified option is set such that the delta for a *call* option is $\delta = N(d_1) = 0.25$ and for a *put* option $\delta = -N(-d_1) = -0.25$ where d_1 is as per Equation (7.56) for the purpose of calculating risk reversals for the modified FX option pricing model.

Empirical Approaches to Risk Reversal Exchange Rate Forecasting

The two most common trading assumptions made by practitioners using risk reversals as a source of information to forecast the change in the spot foreign exchange markets are:

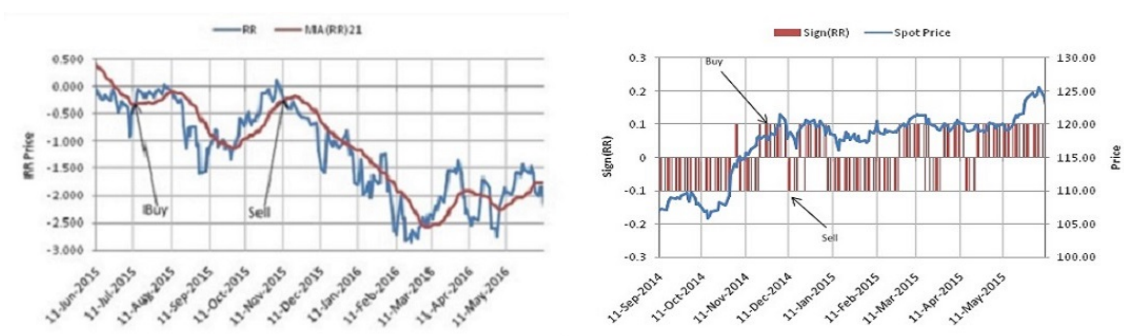
i. Risk reversals are directional trend indicators.

There are two possible strategies where risk reversals are used as a directional indicator of the price:

a. Comparing the 21-day moving average of the risk reversal value to the current risk reversal value. Thereby a buy indicator occurs if the current value cuts the 21-day moving average from below, moving above moving average level. Conversely a sell signal is generated if the current value cuts the 21-day moving average from above, moving below moving average level.

b. Comparing the sign of the risk reversal whereby a negative sign generates a sell signal and a positive sign a buy signal.

FIGURE 7.1: Risk Reversal Direction Trend Indicators



(a) Risk Reversal 21-Day Moving Average (b) Risk Reversal Sign Comparison

ii. Risk reversals are early trend reversal indicators.

This requires devising two mean reverting strategies by separating the series into positive and negative risk reversals. The positive series is used to create an upper band equal to the mean plus one standard deviation. The negative series is used to create a lower band equal to the mean minus one standard deviation. If the risk reversal moves above the upper band this generate a sell signal until the risk reversal moves below the upper band by moving into the neutral zone between the two bands. Conversely a risk reversal below the lower band is a signal to buy until the risk reversal moves above this band into the neutral zone. The mean and standard deviation for each of the strategies are determined by:

a. Estimating the mean and standard deviation on a cumulative basis.

b. Sampling the mean and standard deviations on a rolling 21-day basis.

FIGURE 7.2: Risk Reversal Early Trend Reversal Indicators

(a) Risk Reversal Cumulative Bands (b) Risk Reversal 21-day Moving Average Bands

The returns from these technical trading strategies as applied to the modified FX option pricing model, *see* Equation (7.56), and the BSM model will be compared to a simple 21-day moving average on the spot exchange rate itself. This is one of the oldest and most widely used methods by market practitioners: Dunis (1989). The simple rule states that when the current spot rate penetrates the 21-day moving average from below a buy signal is generated. Conversely if the spot rate penetrates the moving average from above then this generates a sell signal. When comparing the modified FX option pricing model, the BSM model and the 21-day moving average the performance criterion is founded on the premise that the better model will more accurately reflect the information contained in the market and thus more precisely forecast the foreign exchange spot market directional changes.

7.3 Results

Dunis and Lequeux (2001) suggested that market practitioners anticipated a correlation between risk reversals and lagged values of the exchange rate changes. To verify this correlation Dunis and Lequeux (2001) conducted a Granger causality test to determine if a causal link existed between both series. The objective is to investigate the existence of a lagged relationship between the BSM risk reversal series and the FX spot rates series. This relationship underlines the possibility of using the risk reversal to forecast the directional change of foreign exchange rates. The findings of the one-month risk reversals contradicted expectations suggesting there was no causal link, *see* Table 7.1.

TABLE 7.1: Pair-Wise Granger Causality Test: Risk Reversal Do Not Granger Cause Spot Rates

	EUR/GBP	EUR/USD	GBP/USD	USD/JPY	EUR/JPY	GBP/JPY
F-Statistic	0.5804	2.1932	19.4392	0.2267	0.0130	0.6621
Probability	0.5597	0.1118	0.0000	0.7972	0.9871	0.5159
Null Hypothesis:	Accept	Accept	Reject	Accept	Accept	Accept

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From Table 7.1 the Granger causality test does not reject the null hypothesis that risk reversals do not Granger cause spot rates. The Granger test rejects a causal link from the risk reversal to the spot exchange rate changes for all currencies with the exception of GBP/USD. This is in line with expectations with the currency option market being only 15% of the FX spot market: Bank for International Settlements (BIS) 2019. The results are in agreement with the findings of Dunis and Lequeux (2001) in accepting the null hypothesis and rejecting a causal link from the risk reversals to the spot rate changes. The findings for a causal link from the spot exchange rates to the risk reversals however is not as clearly defined; *see* Table 7.2.

TABLE 7.2: Pair-Wise Granger Causality Test: Spot Rates Do Not Granger Cause Risk Reversals

	EUR/GBP	EUR/USD	GBP/USD	USD/JPY	EUR/JPY	GBP/JPY
F-Statistic	0.0291	24.2292	5.4528	1.5152	1.6440	6.4850
Probability	0.9714	0.0000	0.0043	0.2200	0.1934	0.0016
Null Hypothesis:	Accept	Reject	Reject	Accept	Accept	Reject

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The foreign exchange rate is the price of the underlying asset that determines the value of the currency option. The association between the underlying asset price and the option value would suggests a causal link from the spot rate to the risk reversal. From Table 7.2 the Granger causality test does not reject the null hypothesis that spot rates do not Granger cause risk reversals for EUR/GBP, USD/JPY and EUR/JPY. Conversely the Granger test rejects the null hypothesis, suggesting a causal link from the spot exchange rates to the risk reversals for EUR/USD, GBP/USD and GBP/JPY. This causal link suggests that the spot rate does lead the currency option price for EUR/USD, GBP/USD and GBP/JPY as expected, but does not suggest a causal link for EUR/GBP, USD/JPY and EUR/JPY.

However these finding in themselves do not conclusively confirm there is, or is not, embedded information that could assist in determining the future direction of exchange rates and an empirical technical analysis will be undertaken.

Table 7.3 shows the summary statistics of the performance that would have been obtained by using the spot rate 21-day moving average strategy and applying it to the six exchange rates for which the risk reversal strategies are being evaluated.

TABLE 7.3: FX Spot Rates 21-Day Moving Average

	EUR/GBP	EUR/USD	GBP/USD	USD/JPY	EUR/JPY	GBP/JPY
Annualised Return	18.41%	18.57%	18.04%	20.34%	21.80%	22.56%
Annualised Volatility	5.74%	5.70%	5.55%	5.89%	6.94%	6.82%
Return/Volatility Ratio	3.21	3.26	3.25	3.45	3.14	3.31
Maximum Daily Loss	−1.89 %	−2.08 %	−1.52 %	−1.67 %	−2.28 %	−2.60 %
Maximum Daily Profit	6.00%	3.03%	2.98%	3.47%	4.18%	2.78%
Maximum Cumulative Loss	−1.58 %	−3.28 %	−1.96 %	−1.92 %	−2.52 %	−1.57 %
Number of Observations	2,480	2,480	2,480	2,480	2,480	1,792
% Winning Days	29.88%	29.07%	29.44%	30.56%	30.48%	30.47%

The Annualised Return is the cumulative log-return multiplied by 252 trading days over the number of observations. The Annualised Volatility is the standard deviation of the log-returns multiplied by the square root of 252 trading days. The Maximum Daily Loss/Profit is the lowest and highest daily log-return from the sample period. The Maximum Cumulative Loss is the greatest cumulative daily loss for successive periods before reversing the trend. Using a spot rate 21-day moving average, *see* Table 7.3, would have proved a profitable strategy for all six currency pairs during the period investigated. The profitability of the directional trend indicator strategy will be used as a benchmark to gauge the economic value of the four trading strategies based on risk reversals below.

TABLE 7.4: BSM FX Risk Reversals 21-Day Moving Average

	EUR/GBP	EUR/USD	GBP/USD	USD/JPY	EUR/JPY	GBP/JPY
Annualised Return	1.17%	10.94%	6.36%	2.42%	−0.45 %	8.91%
Annualised Volatility	5.88%	6.05%	6.47%	5.95%	6.96%	8.65%
Return/Volatility Ratio	0.20	1.81	0.98	0.41	−0.06	1.03
Maximum Daily Loss	−1.89 %	−1.77 %	−8.40 %	−2.82 %	−2.83 %	-12.27%
Maximum Daily Profit	6.00%	3.03%	2.98%	3.17%	2.59%	3.74%
Maximum Cumulative Loss	−3.21 %	−3.28 %	−2.74 %	−3.30 %	−3.58 %	−3.99 %
Number of Observations	2,480	2,480	2,480	2,480	2,480	1,792
% Winning Days	25.93%	29.23%	27.02%	24.03%	24.88%	28.13%

The results in Table 7.4 show that the BSM risk reversal 21-day moving average strategy has generated significantly lower returns than that achieved by the simple spot rate 21-day moving average across all six exchange rates with EUR/JPY generating a negative return. The results suggest that there is no advantageous lag in the spot prices and the BSM risk reversal rates that can profitably be used within a directional trading strategy. These results are in line with expectations. From the Triennial Survey for the three-year period, 2016-2019 the Bank for International Settlements (BIS) reports that the FX spot market was worth $1.99 trillion a day in 2019 when compared with the currency option market which was only worth $294 million a day in 2019. There is far more information contained in the FX spot market than can be conveyed by the currency option market. Contrast the performance of the BSM risk reversals with the statistics of the alternative directional strategy using the modified risk reversals in Table 7.5 below.

TABLE 7.5: Modified FX Risk Reversals 21-Day Moving Average

	EUR/GBP	EUR/USD	GBP/USD	USD/JPY	EUR/JPY	GBP/JPY
Annualised Return	-14.38%	-13.70%	-18.12%	-16.36%	−2.31 %	-23.58%
Annualised Volatility	5.20%	5.75%	6.32%	6.26%	0.39%	8.89%
Return/Volatility Ratio	−2.76	−2.38	−2.87	−2.61	−5.95	−2.65
Maximum Daily Loss	−2.01 %	−2.41 %	−8.40 %	−3.77 %	−0.69 %	-12.27%
Maximum Daily Profit	1.75%	2.51%	1.90%	2.22%	0.00%	3.74%
Maximum Cumulative Loss	−3.21 %	−3.70 %	−3.16 %	−4.10 %	−0.79 %	−5.11 %
Number of Observations	2,480	2,480	2,480	2,480	2,480	1,792
% Winning Days	19.19%	20.52%	20.24%	19.07%	0.00%	19.14%

Table 7.5 shows that the modified risk reversal 21-day moving average strategy has generated significantly lower returns than the spot rate and the BSM risk reversal 21-day moving average across all six exchange rates. The modified risk reversals returned a negative gain for every exchange rate over the sample period. Although the modified 21-day moving average model appears to perform significantly worse than the BSM 21-day moving average this is a failing of the trading strategy rather than an indication of the performance of the model. The trading strategy is dependent on alternating positive and negative risk reversal values crossing the 21-day moving average to generate a buy and sell signal. This strategy fails and becomes ineffective for prolonged negative values whereby no buy indicators are generated and hence the strategy does not generate a positive return. This problem is further compounded when considering the statistics of the alternative directional strategy whereby the sign actually generates a buy or sell signal, *see* Tables 7.6 and 7.7, which further cast doubts of the potential directional benefits of risk reversals.

TABLE 7.6: BSM FX Risk Reversals - Sign Comparison

	EUR/GBP	EUR/USD	GBP/USD	USD/JPY	EUR/JPY	GBP/JPY
Annualised Return	−0.61 %	0.20%	0.20%	0.57%	−0.13 %	0.18%
Annualised Volatility	5.41%	2.87%	1.76%	3.24%	0.97%	1.97%
Return/Volatility Ratio	−0.11	0.07	0.11	0.18	−0.13	0.09
Maximum Daily Loss	−1.87 %	−1.39 %	−1.57 %	−2.05 %	−1.10 %	−2.08 %
Maximum Daily Profit	1.72%	2.00%	1.92%	1.44%	0.87%	1.35%
Maximum Cumulative Loss	−2.76 %	−1.36 %	−0.90 %	−2.29 %	−1.11 %	−0.90 %
Number of Observations	2,500	2,500	2,500	2,500	2,500	1,812
% Winning Days	20.44%	0.28%	0.48%	1.72%	0.00%	2.37%

Again the BSM risk reversal performance is quite poor, *see* Table 7.6. For the BSM risk reversal sign comparison strategy all the exchange rates underperformed when compared with the BSM risk reversal 21-day moving average strategy and substantially underperformed against the spot rate 21-day moving average strategy with EUR/GBP and EUR/JPY generating negative returns. The poor performance is an indicator of the failing strategy rather than the information contained in the currency option. The failings of the sign comparison strategy are exposed when the risk reversals maintain a negative sign for prolonged periods, thereby voiding a buy signal. This failing is very evident in the modified risk reversal sign comparison strategy, *see* Table 7.7. The statistics highlight a fundamental weakness of the trading strategy that undermines this concept.

TABLE 7.7: Modified FX Risk Reversals - Sign Comparison

	EUR/GBP	EUR/USD	GBP/USD	USD/JPY	EUR/JPY	GBP/JPY
Annualised Return	0.00%	0.00%	0.00%	0.00%	−0.01 %	0.00%
Annualised Volatility	0.00%	0.00%	0.00%	0.00%	0.40%	0.00%
Return/Volatility Ratio	0.00	0.00	0.00	0.00	−0.02	0.00
Maximum Daily Loss	0.00%	0.00%	0.00%	0.00%	−0.69 %	0.00%
Maximum Daily Profit	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Maximum Cumulative Loss	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Number of Observations	2,500	2,500	2,500	2,500	2,500	1,812
% Winning Days	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%

For the modified risk reversal sign comparison strategy, *see* Table 7.7, the results were inconclusive and the strategy ineffective. The strategy reported no returns for EUR/GBP, EUR/USD, GBP/USD, USD/JPY and GBP/JPY due to the risk reversal signs being negative for the entire sample period and negative for a vast majority of the period for EUR/JPY. The prolonged negative signs resulted in no buy indicators from this method and hence no returns were generated. The sign comparison method seems to be highly sensitive to the data that can render it completely ineffective if there are no alternates in the time series. Contrast these strategies with the mean reverting contrarian approach below.

TABLE 7.8: BSM FX Risk Reversals - Cumulative Upper and Lower Bands

	EUR/GBP	EUR/USD	GBP/USD	USD/JPY	EUR/JPY	GBP/JPY
Annualised Return	1.00%	−2.11 %	−2.63 %	0.05%	−2.30 %	−3.88 %
Annualised Volatility	7.46%	11.83%	13.42%	7.34%	9.94%	17.85%
Return/Volatility Ratio	0.13	−0.18	−0.20	0.01	−0.23	−0.22
Maximum Daily Loss	−1.87 %	−0.33 %	0.00%	−0.54 %	0.00%	−0.67 %
Maximum Daily Profit	1.99%	2.06%	2.35%	1.51%	1.98%	3.74%
Maximum Cumulative Loss	−3.49 %	−3.58 %	-11.83%	−3.13 %	−4.33 %	-15.78%
Number of Observations	2,500	2,500	2,500	2,500	2,500	1,812
% Winning Days	38.76%	0.56%	0.00%	0.84%	0.00%	0.83%

Table 7.8 shows that the BSM cumulative upper and lower bands performance is as poor as the sign comparison strategy and significantly worse than the 21-day moving average. All the exchange rates substantially underperformed the moving average method with EUR/USD, GBP/USD, EUR/JPY and GBP/JPY realising negative returns. This was due to prolonged periods of neutral pricing within the bands and lesser periods crossing the upper band trigging a sell indicator and only very infrequent periods crossing the lower band to generate a buy signal. This appears to be a rudimentary trading strategy that is rendered ineffective for prolonged periods of neutral pricing within the upper and lower bands. Compare the performance of the modified cumulative upper and lower bands strategy outlined in Table 7.9.

TABLE 7.9: Modified FX Risk Reversals - Cumulative Upper and Lower Bands

	EUR/GBP	EUR/USD	GBP/USD	USD/JPY	EUR/JPY	GBP/JPY
Annualised Return	1.54%	0.48%	1.58%	4.78%	−0.18 %	6.76%
Annualised Volatility	8.46%	11.39%	5.67%	9.58%	0.31%	9.57%
Return/Volatility Ratio	0.18	0.04	0.28	0.50	−0.58	0.71
Maximum Daily Loss	0.00%	0.00%	0.00%	0.00%	−0.18 %	0.00%
Maximum Daily Profit	1.99%	1.64%	1.33%	3.47%	0.00%	4.16%
Maximum Cumulative Loss	−3.81 %	−3.50 %	−2.11 %	−4.93 %	−0.19 %	−5.06 %
Number of Observations	2,500	2,500	2,500	2,500	2,500	1,812
% Winning Days	0.00%	0.00%	0.00%	0.00%	0.44%	0.00%

The modified cumulative upper and lower bands strategy performed better than the BSM model with EUR/GBP, EUR/USD, GBP/USD, USD/JPY and GBP/JPY generating a positive return and outperforming the modified 21-day moving average. However the strategy performed substantially below that of the spot rate 21-day moving average. It is clear that the FX spot market affects the currency option market raising the question whether the information in the currency option market can drive the spot market price? For the BSM and modified cumulative upper and lower band strategy, *see* Tables 7.8 and 7.9, the data stayed within the neutral bands for prolonged periods negating any buy and sell signals. This strategy resulted in long periods of no trading activity rendering it ineffective.

TABLE 7.10: BSM FX Risk Reversals - 21 Day Moving Average Upper and Lower Bands

	EUR/GBP	EUR/USD	GBP/USD	USD/JPY	EUR/JPY	GBP/JPY
Annualised Return	0.94%	−2.00 %	−2.39 %	0.32%	−2.03 %	−3.94 %
Annualised Volatility	7.46%	11.98%	13.22%	7.27%	9.72%	17.89%
Return/Volatility Ratio	0.13	−0.17	−0.18	0.04	−0.21	−0.22
Maximum Daily Loss	−1.87 %	−0.33 %	0.00%	−0.54 %	0.00%	−0.67 %
Maximum Daily Profit	1.99%	2.06%	2.35%	1.51%	1.98%	3.74%
Maximum Cumulative Loss	−3.49 %	−3.58 %	-11.83%	−3.13 %	−0.18 %	-15.78%
Number of Observations	2,480	2,480	2,480	2,480	2,480	1,792
% Winning Days	38.91%	0.56%	0.00%	0.85%	0.00%	0.73%

From Table 7.10 it can be seen that the poor performance of the BSM 21-day moving average upper and lower bands strategy is in line with results seen in the cumulative upper and lower bands strategy. Both of the upper and lower bands strategies performed as poorly as the sign comparison strategy and significantly worse than the 21-day moving average. All these strategies are highly sensitive to prolonged periods of no alternates in the time series, rendering them ineffective. For the BSM 21-day moving average upper and lower bands, *see* Table 7.10, all the exchange rates substantially underperformed the moving average method with EUR/USD, GBP/USD, EUR/JPY and GBP/JPY realising negative returns. Although the modified 21-day moving average upper and lower bands strategy outperformed the BSM model with EUR/GBP, GBP/USD, USD/JPY and GBP/JPY generating a positive return, *see* Table 7.11, it was still subject to the same frailty in the trading strategy experienced by the cumulative upper and lower bands and the sign comparison strategies, see above.

TABLE 7.11: Modified FX Risk Reversals - 21 Day Moving Average Upper and Lower Bands

	EUR/GBP	EUR/USD	GBP/USD	USD/JPY	EUR/JPY	GBP/JPY
Annualised Return	1.70%	−0.05 %	1.23%	4.74%	−0.19 %	4.77%
Annualised Volatility	8.48%	11.69%	5.56%	9.56%	0.31%	9.18%
Return/Volatility Ratio	0.20	0.00	0.22	0.50	−0.61	0.52
Maximum Daily Loss	0.00%	0.00%	0.00%	0.00%	−0.18 %	0.00%
Maximum Daily Profit	1.99%	1.64%	1.33%	3.47%	0.00%	2.78%
Maximum Cumulative Loss	−3.81 %	−3.50 %	−2.11 %	−4.93 %	−0.20 %	−5.06 %
Number of Observations	2,480	2,480	2,480	2,480	2,480	1,792
% Winning Days	0.00%	0.00%	0.00%	0.00%	0.44%	0.00%

The performance of the modified 21-day moving average upper and lower bands was subject to prolonged periods of neutral pricing within the bands limiting the occasion of triggering a buy or sell signal in the same way that detrimentally effected the BSM version, rendering the outcome inconclusive.

For the BSM pricing model the contrarian strategy was outperformed by both the directional trend strategies and the modified contrarian strategies with the FX spot rate directional indicator using the 21-day moving average generating the highest returns. However, since both of the contrary strategies and the sign comparison directional strategy are dependent on the same fundamentals they are all subject to the same sensitivity in the data that undermines trading concepts.

The trading strategies outlined by Dunis and Lequeux (2001) appear to be insufficient in evaluating the performance of the option pricing models. Although the directional strategies outperformed the contrarian strategies in generating the highest returns for the spot rate 21-day moving average and then the BSM 21-day moving average, the strategies are sensitive to the form of the time series. Data with prolonged negative periods results in no trading activity. The strategies have also resulted in the trend indicators generating lower or even negative returns with periods of no winning days, which is not credible. All the strategies seem to be overly sensitive to the data which at best can render them unreliable and at worst ineffective methods.

Although the modified model outperformed the BSM model in the contrarian strategies this cannot be relied upon as conclusive due to the sensitivity of the trading strategies to the sign of the data. This sensitivity to the sign of the data rendered the trading strategy ineffective.

7.4 Conclusions

This chapter has shown that following the Black and Scholes (1973) methodology and the associated work of Garman and Kohlhagen (1983) under the assumption that the spot price process follows the Kon (1984) parameter shifted modified geometric Brownian motion an alternative FX option pricing model is obtained. This modified FX option pricing model addressed the issues of skewness and kurtosis in the return distribution that limited the performance the market leading BSM pricing model.

This thesis focused on the main variable for which Black-Scholes gives results that differ widely from market data: implied volatility. The implied volatility is the fundamental pricing method for foreign exchange options which are quoted in a significantly different way to other derivatives. The discrepancies observed between market and theoretical prices are pronounced in the different implied volatilities according to the exercise price (smile and skew) and maturities (term structure). The option's relative value can be compared by their implied volatilities. This comparison is of such importance that professional traders often quote the value of options in terms of its implied volatility rather than its premium. The BSM risk reversal options are priced by differencing the implied volatility of a call and put option. The thesis has shown that the risk reversals derived from the alternative FX option pricing model, differencing the price rather than implied volatility of the alternative call and put options, under the assumption that the volatility for the modified risk reversals was the annualised standard deviation for the entire sample period, adjusted for the parameter shift due to the stability of the N normal distributions, yielded an option price significantly different from that of the BSM risk reversals.

The thesis utilised the FX trading strategies employed by practitioners to compare the forecasting performance of the market established BSM derived risk reversals with that of the risk reversals derived from the parameter shifted modified FX model. The performance criterion is founded on the premise that the better model will more accurately reflect the information contained in the market and thus more precisely forecast the foreign exchange spot market directional changes. It was anticipated that the data would show a strong contemporaneous relationship between variations in the risk reversals and the underlying spot series. However the results were in line with expectations and the Granger causality test. Where the strategy was successfully applied the modified FX pricing model eclipsed the performance of the BSM model using the trend reversal strategies. However, the results highlighted a fundamental weakness of the practitioner trading strategies that voided trading indicators from occurring: prolonged periods of negative signs negating a buy signal and conversely prolonged periods of positive signs negating a sell signal. If the time series does not alternate the sign of the risk reversal, this renders the trading method redundant. Consequently both the directional and the contrary trading strategies can be completely undermined by a unidirectional time series.

To examine the integral assumption of the Black and Scholes (1973) formula: that the stochastic price process follows a geometric Brownian motion, this thesis has addressed the known biases and limitations of the influential Black-Scholes model. Rather than append a correction to the BSM model a suitable transformation was identified and applied to the data. This transformation enabled the extrapolation of greater information regarding the distribution of returns. The transformation ensured that the data more closely adhered to the assumption of normality that was fundamental to the BSM derivation. Following the BSM methodology, the derived alternative FX option pricing model was a significant development. This thesis demonstrated the performance enhancements of the alternative option pricing model. With improved transformations and possible further developments regarding a more accurate appraisal of the N normal distributions, this model can be elevated to an even greater extent.

Chapter 8

Conclusion

8.1 Significance

This thesis developed an alternative, closed form, foreign exchange (FX) option pricing model based on the Black and Scholes (1973) and R. C. Merton (1973) (BSM) methodology. Under the assumption that the BSM approach was sufficient in pricing FX options and the known biases and limitations associated with it are due to the data not conforming to the assumption of normality, a suitable transformation of the data was identified that complied with the normality assumption of geometric Brownian motion, on which the BSM model is based.

The thesis added the proviso that market practitioners must be able to use the pricing model in the FX option market. This approach was achieved by dividing the undertaking into three sections, namely:

i. Defining the functionality of the high frequency FX market in order to identify the attributes that explain the FX market spot rates: Chapters 2 and 3.

ii. Deriving a stochastic equation that conforms to the BSM assumption of normality and represents the system of attributes that explain the FX market: Chapters 4 and 5.

iii. Applying the BSM methodology to the fundamental equation describing the FX market in deriving a closed form option pricing model. The alternative FX option pricing model was then evaluated against the market leading BSM model in forecasting the directional movements in the exchange rate: Chapters 6 and 7.

8.2 Findings and Implications

This thesis took the view that market professionals quote prices based on the microstructure variables of order flow and bid-ask spread and the equilibrium restoring triangulation. This view significantly diverted from the established expectations that prices are driven by macroeconomic fundamentals such as inflation and interest rates. Under the assumption that the consequence of market practitioners trading in the FX market defines the market operation, this thesis advocated a microstructure approach, as opposed to a macroeconomic approach, as the appropriate course of action to explain the exchange rate dynamics. Chapter 2 of the thesis demonstrated that the fundamentals of the market, when combined with triangulation, operated as a transmission mechanism conveying information across a network of practitioners that made up the FX market spot price. Consequently the market prices are determined by the circular information flow conveyed from institutional clients to traders, followed by inter-trader transactions, before returning the information flow to the institutional clients. This thesis posits that the transmission mechanisms for dissipating pricing information comprises of the microstructure variables of order flow, bid-ask spread and triangulation acting in concert. This thesis distinctly proposes that each of the elements conveys information unique to its own function; however, collectively they combine to operate as a system of attributes describing the market and arriving at the quoted spot price. Chapter 2 concludes that the FX spot market should be viewed in its entirety as operating as a system of attributes and not taken as separate elements.

Under the proviso that the microstructure approach explains the FX market, the explanatory factors of each of the elements were considered in Chapter 3. Whereas order flow was simply the net buying or selling pressure and triangulation an arbitrage equaliser, the constituents of the bid-ask spread: price discreteness and price clustering required further investigation.

In evaluating the granularity of FX prices the thesis concluded that the price discreteness element of the transmission mechanism was efficient and could not be enhanced by increasing the quote size by an extra digit. The extra level of complexity introduced by an extra digit was found to more than offset any potential benefits. Further, no evidence was found to suggest that increasing the price quotes from five to six digits would impact the size of the spread. The thesis concluded that the information conveyed by price discreteness had been factored into an efficient trading mechanism.

Whereas the test for discreteness was definite, the test for price clustering was not as emphatic. The thesis identified clear evidence of price clustering in the exchange rates considered. However, there was no indication that this was material and was accounted for as a natural occurrence resulting from a compromise between increased accuracy in the price quotes and longer prices. The thesis concluded that even though discreteness and clustering were present in the quotes, these attributes have been assimilated into an efficient information transmission mechanism determining the FX spot price. Thus this thesis advocates an alternative view of the FX market. The thesis proposes that the microstructure attributes of order flow, bid-ask spread and triangulation operate as systemic elements making up the quoting process. Any model of the FX market needs to account for these attributes and their interdependence in defining the information transmission mechanism to explain how the market operates.

Having determined that the FX market is best described as a system of information transmission attributes, Chapter 4 obtained a stochastic price process that represented this system and adhered to the BSM assumption of normality. The thesis examined models that offered an alternative to the normal distribution of returns. Under the condition that the distribution of returns is fat-tailed, the normal distribution is a poor approximation. The thesis identified that the true distribution of returns comprises a symmetrical distribution with fat-tails, higher peaked centre and hollow in-between when compared to a normal distribution. The finite second moment: a well behaved standard deviation, suggests that the distribution will have some properties which non-normal stable distributions do not have. This thesis concluded that a class of fat-tailed distributions with finite second moments will give a better approximation of the distribution of price returns. From the models considered the thesis found that the discrete mixture of normal distributions suggested by Kon (1984) was sufficient to incorporate the system of attributes defining the normal distribution of returns. The model advocated that the actual distribution comprised of N normal distributions, each attributable to exogenous macro and micro information operating as system conveying market information. The thesis concluded that the model could be adapted to describe the information transmission mechanism in modelling the FX market.

Having identified a stochastic pricing model to account for the FX market, Chapter 5 resolved to explain how the combined elements of the transmission mechanism created a skewed and leptokurtic distribution of returns. Under the assumption that if the number of N normal distributions could be identified, the thesis advocated that the stable, modified distribution would be normal by accounting for the number and size of the N shifts in the parameters. This was verified by partitioning the data by year *and* day of the week which returned a normal distribution of returns. The inference drawn from this thesis is that the true mixture of normal distributions is more complex than a simple partition. Notwithstanding this limitation the models espoused by this thesis were sufficient to explain the behaviour and distribution of the foreign exchange market. The thesis concluded that the stochastic price process derived here was a suitable transformation of the data to ensure the modified geometric Brownian motion was sufficient to apply the BSM method in deriving an alternative FX option pricing model.

Under the assumption that the description of the FX market proposed by the discrete mixture of normal distributions was adequate, this thesis proposed an alternative to the standard geometric Brownian motion diffusion process. Chapter 6 modified this diffusion process, which the BSM is based upon, incorporating a parameter shift that transformed the distribution of returns. As a consequence of accounting for the shift variable the distribution was adjusted, adhering to the normality assumption. The Black and Scholes (1973) methodology was then applied to the modified fundamental equation in Chapter 7. This thesis successfully derived an alternative FX option pricing model founded on a discrete mixture of normal distributions. The parameter shifted geometric Brownian motion was the basis of the derivation of an alternative FX option pricing formula that orientated the data around the assumption of normality rather than append to the equation to compensate for the limitations. The modified FX formula was then used to derive risk reversal options, differenced on price, and compared to the market defined BSM risk reversals, differenced by implied volatility. Under the assumption that risk reversals are used by practitioners to forecast the movement of exchange rates the corresponding trading strategies were applied to each model. The performance of the modified FX option model was compared to the market standard BSM model under the proviso the better model will more accurately forecast exchange rate movements. The results were very encouraging. The risk reversals derived from the modified FX option pricing model outperformed the BSM derived risk reversals for both trend reversal strategies, although the performance was not as definite under the direction trend indicator strategies. The thesis concludes that the alternative FX option pricing model is credible and can be used by market practitioners to price FX options. The approach undertaken of transforming the data to adhere to the normality assumption of the BSM model rather than append a correcting feature to it was proper and correct.

8.3 Limitations and Further Extensions

The alternative FX option pricing model proposed by this thesis is a significant step in addressing the known limitations of the BSM formula. However, as for any model, the approach adopted here is a simplified version of reality which is subject to limitations. The first of these is the view that the microstructure variables: order flow, bid-ask spread and triangulation, are the absolute and sufficient determinants in defining the FX market at the exclusion of macroeconomic fundamentals. The reality is likely to be a combination of the two approaches. Thus the system describing the FX market is probably a function of more than the three variables advocated here and is an area that is suitable for supplementary research. Further, when stating that the constituents of the bid-ask spread: price discreteness and price clustering, are elements of an efficient market this is simply the weak form of efficiency pertaining to the information contained in past prices. The transmission mechanism does not address all the publicly available information or all the information affecting the FX spot price.

The second area relates to the explanation offered by the Kon (1984) discrete mixture of normal distribution model for the cause of the skewed and leptokurtic distribution of returns. Simply partitioning the data by year and day of the week to account for normality might appear to be trivial. To identify the true number of N normal distributions will require further interrogation and presents an opportunity for future research.

The third area relates to simplistic view of the information inherent in option prices and how this information is interpreted. The option trading strategies deployed were variants of the spot trading strategies whereas a *forward* trading strategy might have been more appropriate. The fact that options are transacted in the spot market today and mature at some future date written into the contract, they inherently contain information about the market direction. Currency options cannot be used in forecasting exchange rate movements unless the FX option the risk reversal is based upon is priced more accurately. Intuitively a lagged relationship does exist between risk reversals and spot rates, although this invites further research in determining how that relationship can be used to forecast the directional changes of exchange rates. What must follow from this research is to identify how this information can be used in affording a profitable trading opportunity.

Appendix A

Data Set and Detailed Test Statistics

A.1 Variable Definition and Data Set

Abbreviations
EUR Euro
GBP British Pound Sterling
JPY Japanese Yen
USD US Dollar

A.2 Bank for International Settlements (BIS) Data Set

The sample period covers the three months from 12th September 2018 to 12th December 2018 for the top four currencies by volume and respective currency pairs based on the percentage share of average daily turnover in the Bank for International Settlements (BIS) April 2019 survey and consists of EUR/USD, EUR/GBP, GBP/USD, USD/JPY, EUR/JPY and GBP/JPY. It contains the FX limit order price data for the six currency pairs from the Eikon electronic interdealer market. The limit orders comprise the best bid and ask limit orders per second, and time-stamped to the nearest second. No information as to the transaction size or trading parties is given.

A.2.1 Summary Statistics For FX Pip Spreads per Second

TABLE A.1: Currency Pairs - Pip Spread Statistics

	EUR/GBP	EUR/USD	GBP/USD	USD/JPY	EUR/JPY	GBP/JPY
Mean	3.4624	2.7314	3.1789	2.2613	3.7013	3.7853
Median	4.0000	3.0000	4.0000	2.0000	4.0000	5.0000
Maximum	16.0000	11.0000	23.0000	8.0000	8.0000	12.0000
Minimum	1.0000	1.0000	1.0000	0.9000	1.0000	1.0000
Std. Dev.	1.5066	1.3649	1.5535	1.0538	1.3864	1.6611
Skewness	−0.3705	0.1656	−0.1306	0.2227	−0.9773	−0.5892
Kurtosis	2.5560	2.0545	2.3229	2.2992	2.8131	2.0348
Jarque-Bera	2945.2870	4004.7750	2125.1090	2767.6010	15 287.2200	9169.6800
p-value	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Sum	327 917.0000	261 570.0000	307 840.0000	217 833.7000	352 255.0000	359 034.0000
Sum Sq. Dev.	214 985.1000	178 387.7000	233 705.6000	106 969.6000	182 936.8000	261 703.1000
Observations	94,709	95,763	96,838	96,333	95,171	94,850

Eviews 9.
Pip spread per second prices.

A.2.2 Summary Statistics For FX Daily Spot Returns

TABLE A.2: Currency Pairs - Returns Statistics

	EUR/GBP	EUR/USD	GBP/USD	USD/JPY	EUR/JPY	GBP/JPY
Mean	−0.0000	−0.0001	−0.0001	0.0001	−0.0000	0.0000
Median	0.0000	0.0000	−0.0001	0.0000	0.0001	0.0001
Maximum	0.0600	0.0303	0.0298	0.0347	0.0418	0.0416
Minimum	−0.0206	−0.0265	−0.0840	−0.0377	−0.0614	−0.1227
Std. Dev.	0.0051	0.0055	0.0055	0.0056	0.0067	0.0074
Skewness	0.6002	−0.0286	−1.3527	−0.1462	−0.3752	−1.8459
Kurtosis	11.1561	4.9533	24.8917	8.1622	9.1157	35.1536
Jarque-Bera	7385.3320	414.9775	52 873.7000	2905.1230	4125.2630	113 826.9000
p-value	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Sum	−0.0573	−0.2320	−0.1743	0.1871	−0.0447	0.0135
Sum Sq. Dev.	0.0679	0.0789	0.0790	0.0826	0.1183	0.1424
Observations	2,608	2,608	2,608	2,608	2,608	2,608

Eviews 11.
Daily Closing Prices.

A.2.3 Summary Statistics For FX Daily 1-month 25-delta Risk Reversals

TABLE A.3: Currency Pairs - Risk Reversal Statistics

	EUR/GBP	EUR/USD	GBP/USD	USD/JPY	EUR/JPY	GBP/JPY
Mean	0.0623	−0.7213	−0.7577	−0.6264	−1.3412	−1.2749
Median	0.0750	−0.5750	−0.6000	−0.6750	−1.1500	−1.2250
Maximum	7.4250	3.6000	0.6250	0.9880	0.3750	2.3100
Minimum	−3.5500	−4.2380	−9.2750	−2.9750	−6.6500	−9.1000
Std. Dev.	0.8207	0.8475	0.8226	0.6962	0.9696	0.9810
Skewness	2.5299	−1.2981	−5.1366	−0.1165	−2.1918	−3.0421
Kurtosis	24.5806	5.7397	43.7320	2.8470	9.7288	22.7581
Jarque-Bera	51 036.2700	1483.4240	183 745.5000	8.0752	6717.9930	32 268.6200
p-value	0.0000	0.0000	0.0000	0.0176	0.0000	0.0000
Sum	155.4040	−1802.4990	−1893.5790	−1562.9230	−3353.0220	−2310.1440
Sum Sq. Dev.	1678.3680	1794.3290	1690.4200	1208.7510	2349.5230	1742.6570
Observations	2,493	2,499	2,499	2,495	2,500	1,812

Eviews 11.
Daily Closing Price Returns.

A.2.4 Final Digit Distribution Data For The χ^2 Test

TABLE A.4: EUR/GBP χ^2 Test Statistic Data

Final Digit	Observed	Expected	$\frac{(Obs. - Exp.)^2}{Exp.}$
0	9,753	9,471	8.403
1	9,231	9,471	6.077
2	9,353	9,471	1.468
3	8,999	9,471	23.513
4	9,117	9,471	13.224
5	9,441	9,471	0.094
6	9,358	9,471	1.346
7	9,598	9,471	1.706
8	9,677	9,471	4.485
9	10,182	9,471	53.391
		χ^2	113.707
		p-value	0.0000 %

Critical value at the 1% significance level with 9 degrees of freedom is 21.666.

TABLE A.5: EUR/USD χ^2 Test Statistic Data

Final Digit	Observed	Expected	$\frac{(Obs. - Exp.)^2}{Exp.}$
0	9,083	9,576	25.411
1	8,674	9,576	85.017
2	9,453	9,576	1.588
3	9,873	9,576	9.193
4	10,133	9,576	32.363
5	10,215	9,576	42.599
6	9,812	9,576	5.801
7	9,828	9,576	6.616
8	9,642	9,576	0.451
9	9,050	9,576	28.925
		χ^2	237.962
		p-value	0.0000 %

Critical value at the 1% significance level with 9 degrees of freedom is 21.666.

TABLE A.6: GBP/USD χ^2 Test Statistic Data

Final Digit	Observed	Expected	$\frac{(Obs. - Exp.)^2}{Exp.}$
0	9,682	9,684	0.000
1	9,458	9,684	5.265
2	9,701	9,684	0.031
3	9,571	9,684	1.314
4	9,732	9,684	0.240
5	9,626	9,684	0.345
6	9,622	9,684	0.394
7	9,764	9,684	0.664
8	9,969	9,684	8.399
9	9,713	9,684	0.088
		χ^2	16.741
		p-value	5.2931 %

Critical value at the 1% significance level with 9 degrees of freedom is 21.666.

TABLE A.7: USD/JPY χ^2 Test Statistic Data

Final Digit	Observed	Expected	$\frac{(Obs. - Exp.)^2}{Exp.}$
0	10,022	9,633	15.684
1	9,122	9,633	27.138
2	9,529	9,633	1.129
3	9,657	9,633	0.058
4	9,333	9,633	9.361
5	9,418	9,633	4.812
6	9,433	9,633	4.165
7	9,873	9,633	5.964
8	9,829	9,633	3.976
9	10,117	9,633	24.287
		χ^2	96.574
		p-value	0.0000 %

Critical value at the 1% significance level with 9 degrees of freedom is 21.666.

TABLE A.8: EUR/JPY χ^2 Test Statistic Data

Final Digit	Observed	Expected	$\frac{(Obs. - Exp.)^2}{Exp.}$
0	9,467	9,517	0.264
1	9,485	9,517	0.108
2	9,106	9,517	17.758
3	9,403	9,517	1.368
4	9,577	9,517	0.377
5	9,760	9,517	6.199
6	9,773	9,517	6.881
7	9,579	9,517	0.403
8	9,534	9,517	0.030
9	9,487	9,517	0.095
		χ^2	33.483
		p-value	0.0110 %

Critical value at the 1% significance level with 9 degrees of freedom is 21.666.

TABLE A.9: GBP/JPY χ^2 Test Statistic Data

Final Digit	Observed	Expected	$\frac{(Obs. - Exp.)^2}{Exp.}$
0	9,527	9,485	0.186
1	9,071	9,485	18.070
2	9,312	9,485	3.155
3	9,689	9,485	4.388
4	9,401	9,485	0.744
5	9,682	9,485	4.092
6	9,697	9,485	4.738
7	9,184	9,485	9.552
8	9,809	9,485	11.068
9	9,478	9,485	0.005
		χ^2	33.483
		p-value	0.0000 %

Critical value at the 1% significance level with 9 degrees of freedom is 21.666.

A.2.5 Final Digit Distribution Data For The Standardised Range *SR* Test

TABLE A.10: EUR/GBP *SR* Test Statistic Data

Final Digit	Observed	Population	Distribution
0	9,527	94,709	0.1030
1	9,071	94,709	0.0975
2	9,312	94,709	0.0988
3	9,689	94,709	0.0950
4	9,401	94,709	0.0963
5	9,682	94,709	0.0997
6	9,697	94,709	0.0988
7	9,184	94,709	0.1013
8	9,809	94,709	0.1022
9	9,478	94,709	0.1075
		$Hi(\phi_i)$	0.1075
		$Lo(\phi_i)$	0.0950
		SR	0.1249

No clustering returns a value of zero. 100% clustering returns a value of ten.

TABLE A.11: EUR/USD *SR* Test Statistic Data

Final Digit	Observed	Population	Distribution
0	9,083	95,763	0.0948
1	8,674	95,763	0.0906
2	9,453	95,763	0.0987
3	9,873	95,763	0.1031
4	10,133	95,763	0.1058
5	10,215	95,763	0.1067
6	9,812	95,763	0.1025
7	9,828	95,763	0.1026
8	9,642	95,763	0.1007
9	9,050	95,763	0.0945
		$Hi(\phi_i)$	0.1067
		$Lo(\phi_i)$	0.0906
		SR	0.1609

No clustering returns a value of zero. 100% clustering returns a value of ten.

TABLE A.12: GBP/USD *SR* Test Statistic Data

Final Digit	Observed	Population	Distribution
0	9,682	96,838	0.1000
1	9,458	96,838	0.0977
2	9,701	96,838	0.1002
3	9,571	96,838	0.0988
4	9,732	96,838	0.1005
5	9,626	96,838	0.0994
6	9,622	96,838	0.0994
7	9,764	96,838	0.1008
8	9,969	96,838	0.1029
9	9,713	96,838	0.1003
		$Hi(\phi_i)$	0.1029
		$Lo(\phi_i)$	0.0977
		SR	0.0528

No clustering returns a value of zero. 100% clustering returns a value of ten.

TABLE A.13: USD/JPY *SR* Test Statistic Data

Final Digit	Observed	Population	Distribution
0	10,022	96,333	0.1040
1	9,122	96,333	0.0947
2	9,529	96,333	0.0989
3	9,657	96,333	0.1002
4	9,333	96,333	0.0969
5	9,418	96,333	0.0978
6	9,433	96,333	0.0979
7	9,873	96,333	0.1025
8	9,829	96,333	0.1020
9	10,117	96,333	0.1050
		$Hi(\phi_i)$	0.1050
		$Lo(\phi_i)$	0.0947
		SR	0.1033

No clustering returns a value of zero. 100% clustering returns a value of ten.

TABLE A.14: EUR/JPY *SR* Test Statistic Data

Final Digit	Observed	Population	Distribution
0	9,467	95,171	0.0995
1	9,485	95,171	0.0997
2	9,106	95,171	0.0957
3	9,403	95,171	0.0988
4	9,577	95,171	0.1006
5	9,760	95,171	0.1026
6	9,773	95,171	0.1027
7	9,579	95,171	0.1007
8	9,534	95,171	0.1002
9	9,487	95,171	0.0997
		$Hi(\phi_i)$	0.1027
		$Lo(\phi_i)$	0.0957
		SR	0.0701

No clustering returns a value of zero. 100% clustering returns a value of ten.

TABLE A.15: GBP/JPY *SR* Test Statistic Data

Final Digit	Observed	Population	Distribution
0	9,527	94,850	0.1004
1	9,071	94,850	0.0956
2	9,312	94,850	0.0982
3	9,689	94,850	0.1022
4	9,401	94,850	0.0991
5	9,682	94,850	0.1021
6	9,697	94,850	0.1022
7	9,184	94,850	0.0968
8	9,809	94,850	0.1034
9	9,478	94,850	0.0999
		$Hi(\phi_i)$	0.1034
		$Lo(\phi_i)$	0.0956
		SR	0.0778

No clustering returns a value of zero. 100% clustering returns a value of ten.

A.2.6 Attraction Price Clustering Pattern Data

TABLE A.16: EUR/GBP Attraction Price Clustering Pattern Test Statistic Data

Final Digit	Observed	Test Stat.	Result
0	9,753		
1	9,231		
2	9,353		
3	8,999		
4	9,117		
5	9,441		
6	9,358		
7	9,598		
8	9,677		
9	10,182		
Average(Digits 3&7) less Average(Digits 2& 8)	−217		
Average(Digits 2&8) less Average(Digits 4& 6)	278		
Average(Digits 4&6) less Average(Digits 1&9)	−469		
Absolute(Digits 3 less 7)	599		
Absolute(Digits 2 less 8)	324		
Absolute(Digits 4 less 6)	241		
Absolute(Digits 1 less 9)	951		
Minimum(Average)/Average(Absolute)	$\frac{-469}{528.75}$	−0.887	*Reject*

A test statistic greater than one denotes strong evidence in favour of the premiss. Positive values below one suggest the right ordering but a weak fit. The higher a positive number the better the fit and a non-positive value rejects the preposition.

TABLE A.17: EUR/USD Attraction Price Clustering Pattern Test Statistic Data

Final Digit	Observed	Test Stat.	Result
0	9,083		
1	8,674		
2	9,453		
3	9,873		
4	10,133		
5	10,215		
6	9,812		
7	9,828		
8	9,642		
9	9,050		
Average(Digits 3&7) less Average(Digits 2& 8)	303		
Average(Digits 2&8) less Average(Digits 4& 6)	-425		
Average(Digits 4&6) less Average(Digits 1&9)	1,111		
Absolute(Digits 3 less 7)	45		
Absolute(Digits 2 less 8)	189		
Absolute(Digits 4 less 6)	321		
Absolute(Digits 1 less 9)	376		
Minimum(Average)/Average(Absolute)	$\frac{-425}{232.75}$	-1.826	*Reject*

A test statistic greater than one denotes strong evidence in favour of the premiss. Positive values below one suggest the right ordering but a weak fit. The higher a positive number the better the fit and a non-positive value rejects the preposition.

TABLE A.18: GBP/USD Attraction Price Clustering Pattern Test Statistic Data

Final Digit	Observed	Test Stat.	Result
0	9,682		
1	9,458		
2	9,701		
3	9,571		
4	9,732		
5	9,626		
6	9,622		
7	9,764		
8	9,969		
9	9,713		
Average(Digits 3&7) less Average(Digits 2& 8)	−168		
Average(Digits 2&8) less Average(Digits 4& 6)	158		
Average(Digits 4&6) less Average(Digits 1&9)	92		
Absolute(Digits 3 less 7)	193		
Absolute(Digits 2 less 8)	268		
Absolute(Digits 4 less 6)	110		
Absolute(Digits 1 less 9)	255		
Minimum(Average)/Average(Absolute)	$\frac{-168}{206.50}$	−0.811	*Reject*

A test statistic greater than one denotes strong evidence in favour of the premiss. Positive values below one suggest the right ordering but a weak fit. The higher a positive number the better the fit and a non-positive value rejects the preposition.

TABLE A.19: USD/JPY Attraction Price Clustering Pattern Test Statistic Data

Final Digit	Observed	Test Stat.	Result
0	10,022		
1	9,122		
2	9,529		
3	9,657		
4	9,333		
5	9,418		
6	9,433		
7	9,873		
8	9,829		
9	10,117		
Average(Digits 3&7) less Average(Digits 2& 8)	86		
Average(Digits 2&8) less Average(Digits 4& 6)	296		
Average(Digits 4&6) less Average(Digits 1&9)	−237		
Absolute(Digits 3 less 7)	216		
Absolute(Digits 2 less 8)	300		
Absolute(Digits 4 less 6)	100		
Absolute(Digits 1 less 9)	995		
Minimum(Average)/Average(Absolute)	$\frac{-237}{402.75}$	−0.587	*Reject*

A test statistic greater than one denotes strong evidence in favour of the premiss. Positive values below one suggest the right ordering but a weak fit. The higher a positive number the better the fit and a non-positive value rejects the preposition.

TABLE A.20: EUR/JPY Attraction Price Clustering Pattern Test Statistic Data

Final Digit	Observed	Test Stat.	Result
0	9,467		
1	9,485		
2	9,106		
3	9,403		
4	9,577		
5	9,760		
6	9,773		
7	9,579		
8	9,534		
9	9,487		
Average(Digits 3&7) less Average(Digits 2& 8)	171		
Average(Digits 2&8) less Average(Digits 4& 6)	−355		
Average(Digits 4&6) less Average(Digits 1&9)	189		
Absolute(Digits 3 less 7)	176		
Absolute(Digits 2 less 8)	428		
Absolute(Digits 4 less 6)	196		
Absolute(Digits 1 less 9)	2		
Minimum(Average)/Average(Absolute)	$\frac{-355}{200.5}$	−0.587	*Reject*

A test statistic greater than one denotes strong evidence in favour of the premiss. Positive values below one suggest the right ordering but a weak fit. The higher a positive number the better the fit and a non-positive value rejects the preposition.

TABLE A.21: GBP/JPY Attraction Price Clustering Pattern Test Statistic Data

Final Digit	Observed	Test Stat.	Result
0	9,527		
1	9,071		
2	9,312		
3	9,689		
4	9,401		
5	9,682		
6	9,697		
7	9,184		
8	9,809		
9	9,478		
Average(Digits 3&7) less Average(Digits 2& 8)	−124		
Average(Digits 2&8) less Average(Digits 4& 6)	12		
Average(Digits 4&6) less Average(Digits 1&9)	275		
Absolute(Digits 3 less 7)	505		
Absolute(Digits 2 less 8)	497		
Absolute(Digits 4 less 6)	296		
Absolute(Digits 1 less 9)	407		
Minimum(Average)/Average(Absolute)	$\frac{-124}{426.25}$	−0.291	*Reject*

A test statistic greater than one denotes strong evidence in favour of the premiss. Positive values below one suggest the right ordering but a weak fit. The higher a positive number the better the fit and a non-positive value rejects the preposition.

A.2.7 Resolution Price Clustering Pattern Data

TABLE A.22: EUR/GBP Resolution Price Clustering Pattern Test Statistic Data

Final Digit	Observed	Test Stat.	Result
0	9,753		
1	9,231		
2	9,353		
3	8,999		
4	9,117		
5	9,441		
6	9,358		
7	9,598		
8	9,677		
9	10,182		
Average(Digits 2&3&7&8) less Average(Digits 1&4&6&9)	−65		
Maximum(Digits 2or3or7or8) less Minimum(Digits 2or3or7or8)	678		
Maximum(Digits 1or4or6or9) less Minimum(Digits 1or4or6or9)	1,065		
Difference(Average)/Maximum(Maximum-Minimum)	$\frac{-65}{1,065}$	−0.061	*Reject*

A test statistic greater than one denotes strong evidence in favour of the premiss. Positive values below one suggest the right ordering but a weak fit. The higher a positive number the better the fit and a non-positive value rejects the preposition.

TABLE A.23: EUR/USD Resolution Price Clustering Pattern Test Statistic Data

Final Digit	Observed	Test Stat.	Result
0	9,083		
1	8,674		
2	9,453		
3	9,873		
4	10,133		
5	10,215		
6	9,812		
7	9,828		
8	9,642		
9	9,050		
Average(Digits 2&3&7&8) less Average(Digits 1&4&6&9)	282		
Maximum(Digits 2or3or7or8) less Minimum(Digits 2or3or7or8)	420		
Maximum(Digits 1or4or6or9) less Minimum(Digits 1or4or6or9)	1,459		
Difference(Average)/Maximum(Maximum-Minimum)	$\frac{282}{1,459}$	0.193	$WeakFit$

A test statistic greater than one denotes strong evidence in favour of the premiss. Positive values below one suggest the right ordering but a weak fit. The higher a positive number the better the fit and a non-positive value rejects the preposition.

TABLE A.24: GBP/USD Resolution Price Clustering Pattern Test Statistic Data

Final Digit	Observed	Test Stat.	Result
0	9,682		
1	9,458		
2	9,701		
3	9,571		
4	9,732		
5	9,626		
6	9,622		
7	9,764		
8	9,969		
9	9,713		
Average(Digits 2&3&7&8) less Average(Digits 1&4&6&9)	120		
Maximum(Digits 2or3or7or8) less Minimum(Digits 2or3or7or8)	398		
Maximum(Digits 1or4or6or9) less Minimum(Digits 1or4or6or9)	274		
Difference(Average)/Maximum(Maximum-Minimum)	$\frac{120}{398}$	0.302	$WeakFit$

A test statistic greater than one denotes strong evidence in favour of the premiss. Positive values below one suggest the right ordering but a weak fit. The higher a positive number the better the fit and a non-positive value rejects the preposition.

TABLE A.25: USD/JPY Resolution Price Clustering Pattern Test Statistic Data

Final Digit	Observed	Test Stat.	Result
0	10,022		
1	9,122		
2	9,529		
3	9,657		
4	9,333		
5	9,418		
6	9,433		
7	9,873		
8	9,829		
9	10,117		
Average(Digits 2&3&7&8) less Average(Digits 1&4&6&9)	221		
Maximum(Digits 2or3or7or8) less Minimum(Digits 2or3or7or8)	344		
Maximum(Digits 1or4or6or9) less Minimum(Digits 1or4or6or9)	995		
Difference(Average)/Maximum(Maximum-Minimum)	$\frac{221}{995}$	0.222	*WeakFit*

A test statistic greater than one denotes strong evidence in favour of the premiss. Positive values below one suggest the right ordering but a weak fit. The higher a positive number the better the fit and a non-positive value rejects the preposition.

TABLE A.26: EUR/JPY Resolution Price Clustering Pattern Test Statistic Data

Final Digit	Observed	Test Stat.	Result
0	9,467		
1	9,485		
2	9,106		
3	9,403		
4	9,577		
5	9,760		
6	9,773		
7	9,579		
8	9,534		
9	9,487		
Average(Digits 2&3&7&8) less Average(Digits 1&4&6&9)	−175		
Maximum(Digits 2or3or7or8) less Minimum(Digits 2or3or7or8)	473		
Maximum(Digits 1or4or6or9) less Minimum(Digits 1or4or6or9)	288		
Difference(Average)/Maximum(Maximum-Minimum)	$\frac{-175}{473}$	−0.370	*Reject*

A test statistic greater than one denotes strong evidence in favour of the premiss. Positive values below one suggest the right ordering but a weak fit. The higher a positive number the better the fit and a non-positive value rejects the preposition.

TABLE A.27: GBP/JPY Resolution Price Clustering Pattern Test Statistic Data

Final Digit	Observed	Test Stat.	Result
0	9,527		
1	9,071		
2	9,312		
3	9,689		
4	9,401		
5	9,682		
6	9,697		
7	9,184		
8	9,809		
9	9,478		
Average(Digits 2&3&7&8) less Average(Digits 1&4&6&9)	87		
Maximum(Digits 2or3or7or8) less Minimum(Digits 2or3or7or8)	625		
Maximum(Digits 1or4or6or9) less Minimum(Digits 1or4or6or9)	626		
Difference(Average)/Maximum(Maximum-Minimum)	$\frac{87}{626}$	0.139	*WeakFit*

A test statistic greater than one denotes strong evidence in favour of the premiss. Positive values below one suggest the right ordering but a weak fit. The higher a positive number the better the fit and a non-positive value rejects the preposition.

Appendix B

The Properties of Price Processes

B.1 The Properties of Price Processes

B.1.1 Stochastic Processes, Markov Properties and Diffusions

A stochastic process is a random process evolving with time or alternatively, a collection of random variables, $X(t)$, indexed by time, where time is always a subset of non-negative real numbers, $[0, \infty)$. The stochastic process considered here is a continuous time process where a variable can change value at any point in time, such that the random variable $X(t)$ takes any value in the sample space: the set of all possible stock prices or returns, at any point in time.

A Markov process is a particular type of stochastic process where only the present value of a variable is relevant for predicting the future. The past history of the variable and the way that the present has emerged from the past are irrelevant. A process $X(t)$ is Markov if and only if

$$P[X(t)|X(t-1)] = P[X(t)|X(1), \ldots, X(t-1)] \,. \tag{B.1}$$

A Markov process is a class of stochastic process where the change of value of the random variable $X(t)$ at time t is determined by the value of the process at time t and not by the values leading up to t. A diffusion process denotes a continuous time, continuous state space, Markov process whose sample paths are continuous. A model where stock prices change continuously is known as a *diffusion model*. This is referred to as a *mixed jump-diffusion model* when continuous changes are overlaid with jumps and a *pure jump model* when all stock price changes are due to jumps alone. These types of processes are known collectively as *Lévy Processes*.

B.1.2 Brownian Motion

A particular type of Markov process is a Wiener process with a mean equal to zero and a variance equal to the change in the time interval Δt. This is referred to as Brownian motion when describing the motion of a particle subject to a large number of small shocks.

Brownian motion is a stochastic process with both continuous time and continuous sample space that models random continuous motion. A Brownian motion with mean μ and variance σ^2 is a stochastic process where the underlying random variable $X(t)$ at time t can only change continuously and takes real number values satisfying:

i. $X(0) = 0$.

ii. For any $t_n > t_{n-1} > \cdots > t_0 \geq 0$, the random variables $X_{t_k} - X_{t_{k-1}}$ where $k = 1, 2, \ldots, n$ are independent.

iii. For any $\tau > 0$ and $t \geq 0$, the random variable $X(t+\tau) - X(t)$ has a normal distribution with mean $\tau\mu$ and variance $\tau\sigma^2$.

iv. The paths are continuous, that is the function $t \longmapsto X_t$ is a continuous function of t.

Standard Brownian motion is a Brownian motion with $\sigma^2 = 1$.

Expressed formally, a stochastic process $\{z(t), t \geq 0\}$ follows a Wiener process if it exhibits the following properties:

i. The change in value of $z(t)$, Δz, over a small time interval, Δt, follows a normal distribution with a mean 0 and variance equal to the change in the time interval Δt. Specifically

$$\Delta z = z(t + \Delta t) - z(t) \; = \; \epsilon\sqrt{\Delta t}\,, \tag{B.2}$$

where $\epsilon \sim (0, 1)$ is a standardised normal random variable.

ii. The change in the value of $z(t)$, Δz, for any two non-overlapping short intervals of time Δt are independent.

In ordinary calculus it is usual to proceed from small changes to the limit as the small changes become closer to zero. Thus the notation $dx = a\,dt$ is used to indicate $\Delta x = a\,\Delta t$ in the limit as $\Delta t \longrightarrow 0$. Using similar notation in stochastic calculus, where dz refers to a Wiener increment where the change in value of z, Δz, is in the limit as $\Delta t \longrightarrow 0$.

A Wiener process is not differentiable with respect to time, however it is useful to define a term to express the change in the Wiener process with respect to time, dz/dt. A commonly used term for this expression is white noise: a process which has constant mean and variance and zero autocorrelation dependent on the lag and not on time. The white noise process is the derivative of the Wiener process. The standard Wiener process has a drift rate: the rate at which the average changes, of zero and a variance of one with a characteristic function describing the probability distribution of $\sim \phi(0,1)$. The drift rate of zero indicates that the expected value of the underlying random variable $X(t)$ at any future time t is equal to the current value. The variance of one means that the variance of the change in $X(t)$ in a time interval of Δt is equal to Δt.

A generalised Wiener process is obtained by inserting white noise in an ordinary differential equation of the type

$$dx = a\,dt + b\,dz\,, \tag{B.3}$$

where a and b are constants and z is a Wiener process. The first term of Equation (B.3) implies that x has an expected drift rate of a per time unit and the second term involving dz is regarded as adding noise or variability to the path followed by x. The amount of noise is b times the differential of the Wiener process white noise. For a small interval of time the change in the value of x, Δx, is given by

$$\Delta x = a\,\Delta t + b\,\epsilon\sqrt{\Delta t}\,, \tag{B.4}$$

where Δx has a normal distribution with mean $a\,\Delta t$ and variance $b^2\,\Delta t$. Further generalisations of the Wiener process yield the Itô process where the constants a and b are functions of the underlying variable x and time t and which takes the form

$$dx = a\,(x,t)\;dt + b\,(x,t)\;dz \tag{B.5}$$

where dz is a Wiener process. The expected drift rate and variance rate of an Itô process are liable to change over time. This is a Markov process because the change in x at time t depends only on the value of x at time t, not on its history. The variable x has a drift rate of a and a variance of b^2. Itô's lemma shows that a function G of x and t follows the process

$$dG = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right)dt + \frac{\partial G}{\partial x}b\,dz\,, \tag{B.6}$$

where the dz is the same Wiener process as in Equation (B.5). Thus G also follows an Itô process with a drift rate of

$$\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2 \tag{B.7}$$

and a variance rate of

$$\left(\frac{\partial G}{\partial x}\right)^2 b^2\,. \tag{B.8}$$

Wiener Diffusion Process

Any Wiener diffusion process may be considered as the limiting case of a general jump process. The general Wiener diffusion process as a function of the stock price S and time t, (S,t), is given by

$$dS = \mu(S,t)S\,dt + \sigma(S,t)S\,dz\,, \tag{B.9}$$

with drift $\mu(S,t)$ and variance $\sigma^2(S,t)$. Applying the Black and Scholes (1972) assumptions, *see* Section G.1.1 on page 194, leads to the BSM differential equation

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\,\sigma^2(S,t)\,\frac{\partial^2 f}{\partial S^2} = rf\,, \tag{B.10}$$

note that the drift $\mu(S,t)$ does not feature in the BSM differential equation whereas the variance $\sigma^2(S,t)$ does.

B.1.3 Geometric Brownian Motion

Although the drift rate was assumed to be constant in the case of Brownian motion, this is not the situation for stock prices. In the context of stock prices it is not the drift rate that is assumed to be constant but the return on the investment. At a given time t the return on the investment, μ, is specified as the ratio of the drift rate a to the value of stock $S(t)$ at time t; $\mu = a/S(t)$. Hence the constant expected drift rate assumption in Brownian motion is replaced by an assumption of constant expected rate of return in geometric Brownian motion.

Geometric Brownian motion is the stochastic process used to model the evolution of stock prices over time when one believes the percentage change over equal length, non-overlapping intervals are independent and identically distributed (i.i.d.). Let the present time be $t_0 = 0$ where $t_0 < t_1 <, \ldots, < t_n = T$ and $S(t)$ denote the price of the security at times t from the present. Then the collection of prices $S(t)$, $0 \leq t < \infty$ follows a geometric Brownian motion with drift parameter μ and volatility parameter σ if for all non-negative values of t the random variable $S(t+1)/S(t)$ is independent and identically distributed for all prices up to time T.

Let $u_t = S(t+1)/S(t)$ and taking the logarithms of both sides

$$\ln u_t = \ln\left(\frac{S(t+1)}{S(t)}\right) \quad \text{rearranging:} \quad \ln S(t+1) = \ln S(t) + \ln u_t\,, \tag{B.11}$$

if $\ln u_t$ are independent and identically distributed normal random variables with mean μ and variance σ^2 then variable u_t will have a log-normal distribution. The successive prices are found to be an independent product of n log-normal random variables

$$S(t) = S(0)\,u_1\,u_2\,\ldots\,u_n\,, \tag{B.12}$$

taking the natural logarithm gives

$$\ln S(t) = \ln S(0) + \sum_{i=0}^{n} \ln u_i\,, \tag{B.13}$$

where $\ln S(0)$ is constant and each $\ln u_i$ is a normally distributed random variable. Given that the sum of normal variables is a normal variable, then it follows that $\ln S(t)$ is a normal random variable. Hence $S(t)$ has a log-normal distribution with

$$E\left[\ln\left(\frac{S(t)}{S(0)}\right)\right] = \mu t \quad \text{and} \quad Var\left(\ln\left(\frac{S(t)}{S(0)}\right)\right) = \sigma^2 t\,. \tag{B.14}$$

The ratio $\ln(S(t+1)/S(t))$ has a distribution approaching that of a normal random variable with mean μt and variance $\sigma^2 t$. Therefore the series of prices will be a geometric Brownian motion if the ratio of the price at time $S(t+1)$ in the future to the present price $S(t)$ is independent of past price history: a Markov process, and has a log-normal distribution with mean parameter μt and variance parameter $\sigma^2 t$. Note that if $\ln S(t)$ follows a Brownian motion then $S(t)$ follows a geometric Brownian motion since geometric Brownian motion is the exponential of Brownian motion.

Geometric Brownian motion can be formally defined as the variable $S(t)$, $0 \le t < \infty$, with drift parameter μ and volatility parameter σ if for all non-negative values of times t and T the random variable of $S(t+T)/S(t)$ is independent of all values of the variable up to time t and $\ln(S(t+T)/S(t))$ has a normal distribution with mean μt and variance $\sigma^2 t$, independent of t where μ and σ are constants.

To model the stock price evolution let $S(t)$ be the price of stock at time t and assume the expected drift rate $a = \mu S$ for some constant μ. This constant μ is the expected rate of return. The standard deviation of the proportional change in the stock price over a small time interval Δt is $\sigma S\sqrt{\Delta t}$. Thus the stock price can be written as

$$\Delta S = \mu S\,\Delta t + \sigma S\,\Delta z \qquad \text{or} \qquad \frac{\Delta S}{S} = \mu\,\Delta t + \sigma\,\epsilon\sqrt{\Delta t}\,, \tag{B.15}$$

where the Wiener process $\Delta z = \epsilon\sqrt{\Delta t}$ for a short interval of time Δt. In the limit as $\Delta t \longrightarrow 0$ this becomes

$$\frac{dS}{S} = \mu\,dt + \sigma\,\epsilon\sqrt{dt}\,, \tag{B.16}$$

where the variable μ is the stock price expected rate of return and the second term is the stochastic component of the return with σ being the volatility of the stock price. Integrating between time 0 and time t gives

$$E\,[S(t)]) = S(0)\,e^{\left(\mu+\frac{\sigma^2}{2}\right)t}\,. \tag{B.17}$$

Therefore the expected stock price grows exponentially at the rate $\left(\mu + \frac{\sigma^2}{2}\right)$ where $S(0)$ is the initial price at time 0 and where the expected value of the price $S(t)$ at time t depends on both the mean parameter μt and variance parameter $\sigma^2 t$ of the geometric Brownian motion governing the price evolution.

Appendix C

Stable Paretian Distribution Probability Laws

The general characteristic function for the stable distribution is given by the following theorem:

$\phi(u) = e^{\psi(u)}$ is the characteristic function defining the probability distribution of random variable U of a stable law of exponent α, $0 < \alpha < 1$ and $1 < \alpha < 2$ if and only if it has the form

$$\psi(u) = iuc_p - d_p|u|^\alpha \left(1 + i\theta_p \frac{u}{|u|} \tan\left(\frac{\pi}{2}\alpha\right)\right) , \tag{C.1}$$

where the characteristic exponent α is a measures the degree of peakedness that varies from $0 > \alpha \leq 2$ and intimately related to the Pareto exponent θ_p: a real measure of skewness that can vary from $-1 \geq \theta_p \leq +1$, that is $|\theta_p| \leq 1$. If the Pareto exponent $\theta_p = 0$, the stable densities are symmetric. Together α and θ_p determine the type of stable random variable. The shift parameter c_p is real and determines the shift of the distribution, and d_p is real and positive scale factor and i is the median or modal value of U; i has no obvious interpretation when $0 < \alpha < 1$ with $\theta_p \neq 0$.

For $\alpha = 1$, θ_p must vanish, and the form of the characteristic function is given by

$$\psi(u) = iuc_p - d_p|u| \left(1 + i\theta_p \frac{u}{|u|} \frac{2}{\pi} \ln(|u|)\right), \tag{C.2}$$

where c_p, d_p, and θ_p are as above.

The parameters c_p, d_p, and θ_p have useful interpretations. Let the notation $X \triangleq S_\alpha(\sigma, \theta_p, c_p)$ to mean that X is a stable random variable with characteristic exponent α and parameters θ_p, c_p and $d = \sigma^\alpha$.

The parameter c_p is a shift parameter, which can be seen from the fact that if $X \triangleq S_\alpha(\sigma, \theta_p, c_p)$, then $X + a \triangleq S_\alpha(\sigma, \theta_p, c_p + a)$.

To explain the parameter σ, write $\sigma = d^{1/\alpha}$ for a stable distribution with characteristic exponent α. Then σ is said to be a scale parameter because

$$
\begin{aligned}
kX &\triangleq S_\alpha(|k|\sigma, sign(k)\theta_p, kc_p) && \text{if} \quad \alpha \neq 1 \\
kX &\triangleq S_1(|k|\sigma, sign(k)\theta_p, kc_p - \frac{2}{\pi}k(\ln|k|)\sigma\theta_p) && \text{if} \quad \alpha = 1\,.
\end{aligned}
\tag{C.3}
$$

Lastly, Pareto exponent θ_p is said to be a skewness parameter, because $X \triangleq S_\alpha(\sigma, \theta_p, c_p)$ is symmetric if and only if $\theta_p = 0$ and $c_p = 0$. Moreover, X is symmetric about c_p if and only if $\theta_p = 0$.

Appendix D

Lemmas - Compound Events Model

Lemma (1). Let $Z(t)$ be defined as in Equation (4.46) on page 59, and let the $\phi(u) \equiv Ee^{iuZ(t)}$ denote the characteristic function, then:

$$\ln \phi(u) = iCu - \frac{t\sigma_1^2 u^2}{2} + \lambda t \left[e^{i\theta_{ce} u - (\sigma_2^2 u^2/2)} - 1 \right] \tag{D.1}$$

Lemma (2). Let $\phi^*(u)$ denote the characteristics function of the one-step price change $\Delta Z(t) \equiv [Z(t) - Z(t-1]$ defined in Equation (4.47) on page 60, then,

$$\ln \phi^*(u) = -\frac{\sigma_1^2 u^2}{2} + \lambda \left[e^{i\theta_{ce} u - (\sigma_2^2 u^2/2)} - 1 \right] \tag{D.2}$$

Lemma (3). The distribution of $\Delta Z(t)$ is leptokurtic and are the first four cumulants of the distribution of $\Delta Z(t)$

$$\begin{aligned} K_1 &= \lambda\theta_{ce} \,, \\ K_2 &= \sigma_1^2 + \lambda(\theta_{ce}^2 + \sigma_2^2) \,, \\ K_3 &= \lambda\theta_{ce}(\theta_{ce}^2 + 3\,\sigma_2^2) \,, \\ K_4 &= \lambda(\theta_{ce}^4 + 6\,\theta_{ce}^2\sigma_2^2 + 3\,\sigma_2^4) \,, \end{aligned} \tag{D.3}$$

Lemma (4). The distribution of $\Delta Z(t)$ is more peaked than a comparable normal random variable where the density V in the vicinity of the mean is:

$$V \equiv \frac{\Delta Z(t) - E[\Delta Z(t)]}{\sqrt{[var \Delta Z(t)]}} \tag{D.4}$$

Lemma (5). The distribution of $\Delta Z(t)$ is symmetric about its mean if its mean is zero;

otherwise, the distribution is asymmetric. The skewness of the distribution of $\Delta Z(t)$ is defined by:

$$\gamma_1 \equiv \frac{K_3}{\sqrt{K_2^3}} = \frac{\lambda\,\theta_{ce}(\theta_{ce}^2 + 3\,\sigma_2^2)}{\sqrt{[\sigma_1^2 + \lambda\,(\theta_{ce}^2 + \sigma_2^2)]^3}} \tag{D.5}$$

Lemma (6). The density function and cumulative density function of the distribution of $\Delta Z(t)$ may be represented in the form:

$$\begin{aligned} q_{\theta ce}(x) &= \sum_{n=0}^{\infty} \frac{e^{\lambda}\lambda^n}{n!} \times \frac{e^{[(-\frac{1}{2})(x-n\theta_{ce})^2/(n\sigma_2^2+\sigma_1^2)]}}{\sqrt{2\pi(\sigma_1^2+n\sigma_2^2)}} \\ \text{and} & \\ F_{\theta ce}(x) &= \sum_{n=0}^{\infty} \frac{e^{\lambda}\lambda^n}{n!} \times \Phi\left[\frac{x - x\theta_{ce}}{\sqrt{n\sigma_2^2+\sigma_1^2}}\right] \end{aligned} \tag{D.6}$$

where $\Phi(t)$ denotes the cumulative density function of a standard normal variate.

Lemma (7). When $|\theta_{ce}|$ is small, the probability in the extreme tails of the distribution of $\Delta Z(t)$ exceeds that of a comparable normally distributed variable.

Lemma (8). The covariance matrix of the process $Z(t)$ is given by:

$$\psi = \text{diag}[\sigma^2 t + \lambda_1 t({\theta_{ce}}_1^2 + \sigma_1^2), \ldots, \sigma^2 t + \lambda_p t({\theta_{ce}}_p^2 + \sigma_p^2)] + \gamma t(\Sigma + \mu'\mu). \tag{D.7}$$

Lemma (9). The characteristic function of a simple representation the distribution of the vector process $\Delta Z(t)$ is given by:

$$\ln\phi^*(s) = -\frac{\sigma^2}{2}ss' + \gamma[e^{is\mu' - (s\sum s'/2)} - 1] + \sum_{k=1}^{p} \lambda_k[e^{i{\theta_{ce}}_k s_k - (\sigma_k^2 s_k^2/2)} - 1] \tag{D.8}$$

Appendix E

Test for Departure from Normality

E.1 EUR/GBP

E.1.1 Statistics Partitioned by Year

TABLE E.1: EUR/GBP - Test for Departure from Normality

Year	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value[f]	Nbr. of Obs.
Yr. 1	−0.0002	0.0046	−0.4065	−2.6966	2.4541	8.1694	74.0108	0.0000	261
Yr. 2	−0.0001	0.0039	0.4603	3.0534	2.0705	6.8927	56.8322	0.0000	261
Yr. 3	0.0001	0.0048	0.6754	4.4801	0.7917	2.6354	27.0163	0.0000	261
Yr. 4	0.0004	0.0073	1.9833	13.1309	16.7497	55.6535	3269.7287	0.0000	260
Yr. 5	0.0002	0.0062	0.2527	1.6761	−0.0011	−0.0035	2.8094	0.2454	261
Yr. 6	−0.0004	0.0038	−0.2041	−1.3539	0.8955	2.9800	10.7131	0.0047	261
Yr. 7	−0.0002	0.0040	−0.2666	−1.7683	0.5560	1.8508	6.5523	0.0378	261
Yr. 8	0.0001	0.0041	0.0027	0.0181	1.1193	3.7260	13.8833	0.0010	261
Yr. 9	0.0000	0.0053	−0.3261	−2.1632	0.6112	2.0346	8.8191	0.0120	261
Yr. 10	−0.0002	0.0059	−0.0951	−0.6295	−0.3512	−1.1668	1.7578	0.4152	260

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

E.1.2 Statistics Partitioned by Days of The Week

TABLE E.2: EUR/GBP - Test for Departure from Normality

Day of Week	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	0.0001	0.0047	0.2653	2.4820	2.9925	14.0227	202.7954	0.0000	522
Tuesday	0.0000	0.0051	−0.2115	−1.9781	0.9841	4.6112	25.1760	0.0000	522
Wednesday	−0.0005	0.0047	−0.1695	−1.5857	0.5881	2.7560	10.1100	0.0064	522
Thursday	0.0001	0.0053	0.1365	1.2755	0.8391	3.9283	17.0582	0.0002	521
Friday	0.0003	0.0057	2.1209	19.8206	23.4017	109.5537	12 394.8786	0.0000	521

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

E.1.3 Statistics Partitioned by Year and Days of The Week

TABLE E.3: EUR/GBP - Test for Departure from Normality

Year 1	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	0.0012	0.0044	0.5570	1.6858	1.8230	2.8042	10.7054	0.0047	52
Tuesday	−0.0006	0.0047	0.4248	1.2855	2.9399	4.5223	22.1041	0.0000	52
Wednesday	−0.0009	0.0042	−0.8222	−2.5109	3.4482	5.3509	34.9367	0.0000	53
Thursday	−0.0001	0.0044	−1.0820	−3.2746	2.7277	4.1958	28.3275	0.0000	52
Friday	−0.0004	0.0050	−1.1324	−3.4271	2.1864	3.3632	23.0563	0.0000	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.4: EUR/GBP - Test for Departure from Normality

Year 2	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value[f]	Nbr. of Obs.
Monday	0.0005	0.0034	0.6295	1.9051	0.2607	0.4010	3.7903	0.1503	52
Tuesday	−0.0007	0.0033	0.2413	0.7368	−0.3057	−0.4743	0.7679	0.6812	53
Wednesday	−0.0003	0.0034	−0.0866	−0.2622	1.7353	2.6693	7.1937	0.0274	52
Thursday	0.0002	0.0048	0.5559	1.6823	2.7944	4.2984	21.3065	0.0000	52
Friday	−0.0001	0.0043	0.4757	1.4396	1.3767	2.1177	6.5572	0.0377	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.5: EUR/GBP - Test for Departure from Normality

Year 3	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value[f]	Nbr. of Obs.
Monday	0.0002	0.0044	0.4849	1.4807	0.4795	0.7440	2.7461	0.2533	53
Tuesday	0.0001	0.0046	−0.3963	−1.1994	0.5115	0.7868	2.0576	0.3574	52
Wednesday	−0.0005	0.0036	−0.6765	−2.0475	−0.2389	−0.3674	1.3274	0.1149	52
Thursday	−0.0004	0.0056	0.7123	2.1558	1.6915	2.6019	11.4173	0.0033	52
Friday	0.0012	0.0053	−0.1919	−0.5809	0.1064	0.1637	0.3642	0.8335	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.6: EUR/GBP - Test for Departure from Normality

Year 4	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value[f]	Nbr. of Obs.
Monday	0.0002	0.0065	0.7718	2.3359	5.4068	8.3170	74.6288	0.0000	52
Tuesday	−0.0001	0.0073	−0.5393	−1.6322	0.4239	0.6521	3.0891	0.2134	52
Wednesday	−0.0005	0.0059	−0.5769	−1.7460	0.4529	0.6967	3.5339	0.1709	52
Thursday	−0.0001	0.0058	−0.1731	−0.5239	0.0866	0.1332	0.2922	0.8641	52
Friday	0.0025	0.0101	3.6978	11.1915	20.2867	21.2058	1099.0524	0.0000	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.7: EUR/GBP - Test for Departure from Normality

Year 5	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	−0.0003	0.0058	0.2529	0.7655	0.3278	0.5043	0.8403	0.6569	52
Tuesday	0.0002	0.0060	0.0216	0.0653	−0.8895	−1.3682	1.8763	0.3914	52
Wednesday	−0.0001	0.0063	0.5161	1.5619	−0.8200	−1.2614	4.0306	0.1333	52
Thursday	0.0011	0.0072	0.2126	0.6433	0.1149	0.1767	0.4451	0.8005	52
Friday	−0.0001	0.0057	0.0261	0.0798	1.5081	2.3402	5.4831	0.0645	53

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.8: EUR/GBP - Test for Departure from Normality

Year 6	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	0.0002	0.0033	0.7490	2.2670	0.6527	1.0041	6.1475	0.0462	52
Tuesday	−0.0001	0.0033	0.3015	0.9124	0.0026	0.0041	0.8325	0.6595	52
Wednesday	−0.0011	0.0041	−0.3464	−1.0484	0.5850	0.8999	1.9090	0.3850	52
Thursday	−0.0009	0.0041	−0.3031	−0.9257	1.0776	1.6722	3.6533	0.1610	53
Friday	−0.0003	0.0041	−0.4807	−1.4548	1.0429	1.6043	4.6903	0.0958	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.9: EUR/GBP - Test for Departure from Normality

Year 7	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	−0.0002	0.0036	0.2994	0.9062	0.7855	1.2083	2.2812	0.3196	52
Tuesday	0.0009	0.0036	−0.1140	−0.3450	0.0998	0.1535	0.1426	0.9312	52
Wednesday	−0.0012	0.0043	−0.5955	−1.8187	0.3005	0.4663	3.5251	0.1716	53
Thursday	−0.0009	0.0049	0.1302	0.3939	0.1110	0.1708	0.1843	0.9120	52
Friday	0.0004	0.0032	−0.5093	−1.5413	1.2537	1.9285	6.0949	0.0475	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.10: EUR/GBP - Test for Departure from Normality

Year 8	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	0.0000	0.0042	−0.4679	−1.4160	2.1565	3.3172	13.0093	0.0015	52
Tuesday	0.0001	0.0037	0.1365	0.4170	1.2817	1.9889	4.1298	0.1268	53
Wednesday	0.0000	0.0039	−0.1249	−0.3780	0.7087	1.0901	1.3313	0.5139	52
Thursday	0.0000	0.0041	−0.4494	−1.3600	0.5181	0.7970	2.4849	0.2887	52
Friday	0.0007	0.0046	−0.5770	1.7464	0.8596	1.3222	4.7983	0.0908	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.11: EUR/GBP - Test for Departure from Normality

Year 9	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	−0.0005	0.0050	−0.4991	−1.5244	2.7439	4.2580	20.4540	0.0000	53
Tuesday	0.0010	0.0049	0.1025	0.3101	−0.0887	−0.1364	0.1148	0.9442	52
Wednesday	−0.0003	0.0047	−0.5558	−1.6821	0.0486	0.0748	2.8350	0.2423	52
Thursday	0.0012	0.0055	−0.1593	−0.4822	0.6215	0.9560	1.1464	0.5637	52
Friday	−0.0016	0.0058	−0.4623	−1.3990	−0.3821	−0.5877	2.3028	0.3162	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.12: EUR/GBP - Test for Departure from Normality

Year 10	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	0.0004	0.0053	−0.2505	−0.7580	−0.5439	−0.8367	1.2747	0.5287	52
Tuesday	−0.0009	0.0073	−0.1232	−0.3730	−0.5566	−0.8562	0.8722	0.6466	52
Wednesday	−0.0002	0.0056	0.0097	0.0293	−0.5305	−0.8160	0.6667	0.7165	52
Thursday	0.0007	0.0057	−0.0045	−0.0138	−0.7474	−1.1497	1.3220	0.5163	52
Friday	0.0004	0.0053	−0.2505	−0.7580	−0.5439	−0.8367	1.2747	0.5287	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

E.2 EUR/USD

E.2.1 Statistics Partitioned by Year

TABLE E.13: EUR/USD - Test for Departure from Normality

Year	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Yr. 1	−0.0002	0.0029	−0.0193	−0.1283	0.5594	1.8621	3.4839	0.1752	261
Yr. 2	−0.0003	0.0043	−0.2371	−1.5727	0.6866	2.2855	7.6973	0.0213	261
Yr. 3	0.0006	0.0046	−0.0299	−0.1982	0.2025	0.6742	0.4938	0.7812	261
Yr. 4	−0.0002	0.0050	−0.1315	−0.8704	2.2885	7.6038	58.5749	0.0000	260
Yr. 5	−0.0001	0.0074	0.3532	2.3431	1.2940	4.3075	24.0443	0.0000	261
Yr. 6	−0.0007	0.0045	−0.5001	−3.3171	2.8240	9.4008	99.3783	0.0000	261
Yr. 7	0.0001	0.0044	0.0499	0.3308	0.4909	1.6341	2.7798	0.2491	261
Yr. 8	0.0000	0.0050	0.2997	1.9879	0.7963	2.6510	10.9794	0.0041	261
Yr. 9	−0.0001	0.0073	−0.3278	−2.1747	0.3232	1.0760	5.8870	0.0527	261
Yr. 10	0.0000	0.0047	−0.1379	−0.9129	−0.3865	1.2842	2.4826	0.2890	260

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

E.2.2 Statistics Partitioned by Days of The Week

TABLE E.14: EUR/USD - Test for Departure from Normality

Day of Week	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	−0.0003	0.0047	−0.1363	−1.2749	2.7196	12.7439	164.0333	0.0000	522
Tuesday	−0.0002	0.0051	−0.0174	−0.1631	1.3401	6.2796	39.4606	0.0000	522
Wednesday	−0.0001	0.0058	−0.0767	−0.7178	2.1525	10.0864	102.2508	0.0000	522
Thursday	0.0003	0.0061	0.0344	0.3216	2.1233	9.9399	98.9053	0.0000	521
Friday	−0.0001	0.0057	−0.0724	−0.6764	1.0055	4.7073	22.6165	0.0000	521

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

E.2.3 Statistics Partitioned by Year and Days of The Week

TABLE E.15: EUR/USD - Test for Departure from Normality

Year 1	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	0.0001	0.0028	0.1252	0.3789	1.5508	2.3856	5.8344	0.0541	52
Tuesday	−0.0005	0.0024	0.1768	0.5351	0.2290	0.3523	0.4104	0.8145	52
Wednesday	−0.0005	0.0029	−0.1925	−0.5880	0.0969	0.1504	0.3684	0.8319	53
Thursday	−0.0003	0.0028	−0.3843	−1.1632	2.1701	3.3381	12.4958	0.0019	52
Friday	0.0004	0.0033	−0.0162	−0.0489	−0.5552	−0.8540	0.7317	0.6936	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.16: EUR/USD - Test for Departure from Normality

Year 2	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	0.0006	0.0034	−0.6939	−2.1000	0.8997	1.3839	6.3251	0.0423	52
Tuesday	−0.0013	0.0038	0.2433	0.7432	−0.3543	−0.5498	0.8548	0.6523	53
Wednesday	−0.0005	0.0045	0.1436	0.4347	−0.0138	−0.0212	0.1894	0.9096	52
Thursday	−0.0004	0.0052	−0.8252	−2.4975	1.8211	2.8013	14.0852	0.0009	52
Friday	−0.0001	0.0045	0.2619	0.7925	−0.2816	−0.4332	0.8158	0.6650	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.17: EUR/USD - Test for Departure from Normality

Year 3	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	−0.0001	0.0042	0.2826	0.8631	0.7769	1.2055	2.1983	0.3332	53
Tuesday	0.0008	0.0045	0.7539	2.2818	0.4726	0.7270	5.7351	0.0568	52
Wednesday	0.0006	0.0047	0.0728	0.2204	−0.2178	−0.3351	0.1608	0.9227	52
Thursday	0.0006	0.0047	−0.3513	−1.0633	0.4526	0.6962	1.6153	0.4459	52
Friday	0.0011	0.0047	0.4223	1.2779	0.0034	0.0052	1.6332	0.4419	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.18: EUR/USD - Test for Departure from Normality

Year 4	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	−0.0002	0.0042	−0.5493	−1.6623	0.3640	0.5598	3.0768	0.2147	52
Tuesday	−0.0004	0.0044	0.7505	2.2713	0.4055	0.6238	5.5479	0.0624	52
Wednesday	0.0001	0.0047	−0.4674	−1.4146	−0.2090	−0.3215	2.1046	0.3491	52
Thursday	0.0004	0.0054	0.2038	0.6168	0.0938	1.4427	2.4619	0.2920	52
Friday	−0.0009	0.0062	−0.3176	−0.9613	4.1403	6.3688	41.4861	0.0000	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.19: EUR/USD - Test for Departure from Normality

Year 5	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	−0.0005	0.0064	0.7675	2.3228	1.7597	2.7069	12.7228	0.0017	52
Tuesday	−0.0008	0.0069	0.2348	0.7106	0.8271	1.2723	2.1238	0.3458	52
Wednesday	0.0013	0.0080	0.3178	0.9619	0.8442	1.2986	2.6116	0.2710	52
Thursday	0.0006	0.0088	0.2514	0.7608	1.7697	2.7222	7.9891	0.0184	52
Friday	−0.0010	0.0067	−0.0422	−0.1288	0.1118	0.1735	0.0467	0.9769	53

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.20: EUR/USD - Test for Departure from Normality

Year 6	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	0.0002	0.0033	0.8221	2.4880	2.1062	3.2399	16.6873	0.0002	52
Tuesday	0.0001	0.0042	0.8116	2.4562	1.2477	1.9193	9.7164	0.0078	52
Wednesday	−0.0011	0.0047	0.0529	0.1602	1.7343	2.6678	7.1426	0.0281	52
Thursday	−0.0011	0.0053	−1.2903	−3.9404	4.5412	7.0469	65.1859	0.0000	53
Friday	−0.0017	0.0048	−0.8365	−2.5316	0.5646	0.8685	7.1633	0.0278	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.21: EUR/USD - Test for Departure from Normality

Year 7	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	0.0003	0.0035	−1.2683	−3.8384	2.1824	3.3571	26.0031	0.0000	52
Tuesday	0.0004	0.0037	0.2686	0.8130	0.3129	0.4812	0.8925	0.6400	52
Wednesday	−0.0003	0.0049	0.5230	1.5972	1.3923	2.1606	7.2191	0.0271	53
Thursday	0.0005	0.0058	−0.0112	−0.0339	−0.7437	−1.1440	1.3100	0.5194	52
Friday	−0.0004	0.0037	−0.1304	−0.3946	−0.5820	−0.8953	0.9572	0.6197	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.22: EUR/USD - Test for Departure from Normality

Year 8	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	−0.0003	0.0036	−0.2621	−0.7931	−0.5040	−0.7753	1.2302	0.5406	52
Tuesday	−0.0001	0.0046	0.0718	0.2194	−0.4192	−0.6506	0.4714	0.7900	53
Wednesday	−0.0007	0.0042	−0.1389	−0.4203	0.4583	0.7050	0.6737	0.7140	52
Thursday	0.0002	0.0057	0.1001	0.3030	0.3064	0.4713	0.3139	0.8547	52
Friday	0.0011	0.0064	0.4047	1.2248	0.3623	0.5574	1.8109	0.4044	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.23: EUR/USD - Test for Departure from Normality

Year 9	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	−0.0014	0.0073	−0.2402	−0.7335	1.1942	1.8531	3.9720	0.1372	53
Tuesday	0.0013	0.0056	−0.1209	−0.3659	−0.0091	−0.0140	0.1341	0.9351	52
Wednesday	−0.0011	0.0080	−0.3346	−1.0126	−0.1856	−0.2855	1.1068	0.5750	52
Thursday	0.0010	0.0074	−0.4594	−1.3905	1.1188	1.7210	4.8952	0.0865	52
Friday	−0.0004	0.0076	−0.1251	−0.3787	−0.5202	−0.8002	0.7837	0.6758	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.24: EUR/USD - Test for Departure from Normality

Year 10	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	0.0006	0.0070	−0.1577	−0.4774	−0.5576	−0.8577	0.9635	0.6177	52
Tuesday	−0.0015	0.0083	−0.2603	−0.7877	−0.5268	−0.8103	1.2771	0.5281	52
Wednesday	0.0008	0.0084	−0.6930	−2.0975	1.6600	2.5534	10.9194	0.0043	52
Thursday	0.0014	0.0078	0.3393	1.0270	0.2477	0.3810	1.1998	0.5489	52
Friday	0.0006	0.0070	−0.1577	−0.4774	−0.5576	−0.8577	0.9635	0.6177	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

E.3 GBP/USD

E.3.1 Statistics Partitioned by Year

TABLE E.25: GBP/USD - Test for Departure from Normality

Year	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Yr. 1	0.0000	0.0050	0.6702	4.4454	2.3706	7.8916	82.0393	0.0000	261
Yr. 2	−0.0003	0.0050	0.0272	0.1804	0.7323	2.4377	5.9751	0.0504	261
Yr. 3	0.0005	0.0051	−1.6514	−10.9542	1.4440	4.8068	143.0994	0.0000	261
Yr. 4	−0.0006	0.0090	−3.1197	−20.6544	28.8453	95.8429	9612.4595	0.0000	260
Yr. 5	−0.0003	0.0054	0.1996	1.3241	0.5332	1.7750	4.9039	0.2454	261
Yr. 6	−0.0003	0.0037	−0.4079	−2.7059	1.7606	5.8608	41.6708	0.0000	261
Yr. 7	0.0003	0.0046	0.0558	0.3699	0.6497	2.1628	4.8146	0.0901	261
Yr. 8	−0.0001	0.0040	0.0371	0.2458	0.4307	1.4337	2.1160	0.3471	261
Yr. 9	−0.0001	0.0051	−0.2280	−1.5123	−0.0997	−0.3320	2.3974	0.3016	261
Yr. 10	0.0002	0.0030	−0.2845	−1.8835	0.3221	1.0701	4.6928	0.0957	260

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

E.3.2 Statistics Partitioned by Days of The Week

TABLE E.26: GBP/USD - Test for Departure from Normality

Day of Week	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value[f]	Nbr. of Obs.
Monday	−0.0003	0.0048	−0.6230	−5.8280	6.0755	28.4693	844.4643	0.0000	522
Tuesday	−0.0002	0.0053	0.3359	3.1420	2.7077	12.6878	170.8533	0.0000	522
Wednesday	0.0004	0.0053	0.0097	0.0907	0.7921	3.7117	13.7850	0.0010	522
Thursday	0.0002	0.0054	0.0812	0.7591	1.3767	6.4449	42.1129	0.0000	521
Friday	−0.0004	0.0066	−3.9576	−36.9848	49.7516	232.9091	55 614.5103	0.0000	521

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

E.3.3 Statistics Partitioned by Year and Days of The Week

TABLE E.27: GBP/USD - Test for Departure from Normality

Year 1	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value[f]	Nbr. of Obs.
Monday	−0.0010	0.0046	−0.5101	−1.5437	1.7234	2.6509	9.4104	0.0090	52
Tuesday	0.0001	0.0050	0.1866	0.5648	1.9118	2.9408	8.9672	0.0113	52
Wednesday	0.0004	0.0048	1.5004	4.5821	5.3491	8.3006	89.8958	0.0000	53
Thursday	−0.0001	0.0051	0.9991	3.0239	2.9568	4.5483	29.8306	0.0000	52
Friday	0.0005	0.0054	0.9526	2.8829	0.7955	1.2236	9.8084	0.0074	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.28: GBP/USD - Test for Departure from Normality

Year 2	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	0.0001	0.0046	−0.5964	−1.8049	0.3710	0.5707	3.5836	0.1667	52
Tuesday	−0.0005	0.0047	0.1224	0.3739	−0.0738	−0.1145	0.1529	0.9264	53
Wednesday	−0.0003	0.0051	0.2205	0.6673	0.3187	0.4903	0.6856	0.7098	52
Thursday	−0.0006	0.0057	0.3665	1.1093	2.4072	3.7028	14.9413	0.0006	52
Friday	0.0001	0.0052	−0.1823	−0.5516	0.0834	0.1283	0.3207	0.8518	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.29: GBP/USD - Test for Departure from Normality

Year 3	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	−0.0006	0.0045	−0.0167	−0.0509	−0.1715	−0.2661	0.0734	0.9640	53
Tuesday	0.0007	0.0053	1.2348	3.7371	3.6480	5.6114	45.4543	0.0000	52
Wednesday	0.0012	0.0047	0.8624	2.6101	1.4565	2.2404	11.8320	0.0027	52
Thursday	0.0009	0.0050	−0.4268	−1.2918	1.2399	1.9073	2.3066	0.0704	52
Friday	0.0001	0.0060	−0.0389	−0.1176	0.5979	0.9197	0.8596	0.6506	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.30: GBP/USD - Test for Departure from Normality

Year 4	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	−0.0006	0.0084	−0.9101	−2.7543	4.8513	7.4625	63.2747	0.0000	52
Tuesday	−0.0001	0.0093	0.4835	1.4634	1.2459	1.9164	5.8143	0.0546	52
Wednesday	0.0006	0.0051	−0.2389	−0.7229	−0.5324	−0.8189	1.1932	0.5507	52
Thursday	0.0004	0.0068	0.1678	0.5077	0.1518	0.2336	0.3123	0.8554	52
Friday	−0.0031	0.0131	−4.7747	−14.4508	29.2578	45.0055	2234.3176	0.0000	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.31: GBP/USD - Test for Departure from Normality

Year 5	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value[f]	Nbr. of Obs.
Monday	0.0000	0.0041	0.8440	2.5544	0.5002	0.7694	7.1169	0.0285	52
Tuesday	−0.0011	0.0051	0.3384	1.0241	−0.6860	−1.0553	2.1625	0.3392	52
Wednesday	0.0014	0.0063	0.0941	0.2848	0.3407	0.5241	0.3558	0.8370	52
Thursday	−0.0005	0.0050	−0.2879	−0.8713	1.5084	2.3203	6.1429	0.0464	52
Friday	−0.0011	0.0060	0.1519	0.4638	0.3706	0.5751	0.5459	0.7611	53

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.32: GBP/USD - Test for Departure from Normality

Year 6	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value[f]	Nbr. of Obs.
Monday	−0.0001	0.0034	−0.9899	−2.9960	4.4069	6.7789	54.9293	0.0000	52
Tuesday	0.0001	0.0039	−0.2303	−0.6969	1.0005	1.5391	2.8544	0.2400	52
Wednesday	0.0000	0.0038	−0.6146	1.8601	0.5832	0.8970	4.2647	0.1186	52
Thursday	−0.0002	0.0039	0.4638	1.4165	1.3017	2.0199	6.0865	0.0477	53
Friday	−0.0013	0.0034	−1.4736	−4.4597	3.8438	5.9127	54.8493	0.0000	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.33: GBP/USD - Test for Departure from Normality

Year 7	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value[f]	Nbr. of Obs.
Monday	0.0005	0.0034	−1.1085	−0.3284	−0.0120	−0.0184	0.1082	0.9473	52
Tuesday	−0.0004	0.0034	−0.3323	−1.0057	−0.5787	−0.8902	1.8038	0.4058	52
Wednesday	0.0008	0.0055	−0.0451	−0.1376	0.2399	0.3723	0.1576	0.9242	53
Thursday	0.0014	0.0053	0.0462	0.1399	0.6276	0.9654	0.9516	0.6214	52
Friday	−0.0009	0.0046	−0.0545	−0.1648	0.0430	0.0661	0.0315	0.9844	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.34: GBP/USD - Test for Departure from Normality

Year 8	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	−0.0003	0.0032	−0.4516	−1.3666	0.2466	0.3794	2.0116	0.3658	52
Tuesday	−0.0002	0.0032	−0.3718	−1.1354	0.4185	0.6494	1.7109	0.4251	53
Wednesday	−0.0006	0.0037	−0.4270	−1.2923	0.5343	0.8219	2.3455	0.3095	52
Thursday	0.0003	0.0043	0.1176	0.3558	0.3584	0.5512	0.4305	0.8063	52
Friday	0.0004	0.0054	0.1181	0.3574	−0.5414	−0.8327	0.8212	0.6633	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.35: GBP/USD - Test for Departure from Normality

Year 9	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	−0.0009	0.0046	−0.0141	−0.0432	−0.2431	−0.3773	0.1442	0.9304	53
Tuesday	0.0003	0.0050	−0.1697	−0.5135	−0.2986	−0.4593	0.4746	0.7888	52
Wednesday	−0.0008	0.0066	−0.2972	−0.8995	−0.4325	−0.6653	1.2518	0.5348	52
Thursday	−0.0001	0.0048	−0.0524	−0.1585	−0.3178	−0.4889	0.2642	0.8763	52
Friday	0.0011	0.0044	0.0523	0.1587	−0.6116	−0.0941	0.9102	0.6344	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.36: GBP/USD - Test for Departure from Normality

Year 10	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	0.0002	0.0022	−0.2378	−0.7198	−0.6340	−0.9752	1.4693	0.4797	52
Tuesday	−0.0005	0.0063	−0.2096	−0.6344	−0.5951	−0.9155	1.2406	0.5378	52
Wednesday	0.0010	0.0062	−0.7096	−2.1477	0.4045	0.6222	4.9998	0.0821	52
Thursday	0.0007	0.0071	−0.3672	−1.1112	1.3988	2.1517	5.8648	0.0533	52
Friday	0.0002	0.0071	−0.2378	−0.7198	−0.6340	−0.9752	1.4692	0.4797	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

E.4 USD/JPY

E.4.1 Statistics Partitioned by Year

TABLE E.37: USD/JPY - Test for Departure from Normality

Year	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Yr. 1	0.0000	0.0033	−0.3123	−2.0714	2.0026	6.6665	48.7331	0.0000	261
Yr. 2	0.0001	0.0038	−0.0048	−0.0318	0.5692	1.8947	3.5909	0.1661	261
Yr. 3	−0.0002	0.0047	−0.1427	−0.9468	1.1561	3.8485	15.7076	0.0004	261
Yr. 4	0.0000	0.0079	−0.6874	−4.5510	3.1559	10.4860	130.6683	0.0000	260
Yr. 5	−0.0002	0.0052	−0.9289	−6.1612	4.6360	15.4328	276.1332	0.0000	261
Yr. 6	0.0006	0.0051	0.6808	4.5156	3.9582	13.1766	194.0132	0.0000	261
Yr. 7	0.0003	0.0073	0.0871	0.5778	2.3008	7.6591	58.9953	0.0000	261
Yr. 8	0.0006	0.0052	0.4336	2.8763	0.6525	2.1720	12.9904	0.0015	261
Yr. 9	−0.0002	0.0057	0.3734	2.4771	8.9146	29.6760	8886.8038	0.0000	261
Yr. 10	−0.0004	0.0064	−0.0063	−0.0419	6.0722	20.1758	407.0650	0.0000	260

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

E.4.2 Statistics Partitioned by Days of The Week

TABLE E.38: USD/JPY - Test for Departure from Normality

Day of Week	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	0.0000	0.0051	0.3316	3.1017	6.3008	29.5248	881.3361	0.0000	522
Tuesday	0.0002	0.0052	−0.4167	−0.8981	2.1573	10.1088	117.3826	0.0000	522
Wednesday	0.0003	0.0057	−0.0317	−0.2963	4.3373	20.3239	413.1507	0.0000	522
Thursday	−0.0001	0.0061	−0.1735	−1.6218	6.2583	29.2980	861.0059	0.0000	521
Friday	−0.0001	0.0060	−0.3173	−2.9656	5.5031	25.7626	672.5080	0.0000	521

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

E.4.3 Statistics Partitioned by Year and Days of The Week

TABLE E.39: USD/JPY - Test for Departure from Normality

Year 1	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	0.0040	0.0027	−0.3523	−1.0663	0.3806	0.5855	1.4798	0.4772	52
Tuesday	0.0002	0.0034	1.3310	4.0282	3.4507	5.3808	44.4010	0.0000	52
Wednesday	0.0005	0.0030	−0.2763	−0.8439	0.3656	0.5673	1.0340	0.5963	53
Thursday	−0.0003	0.0038	−0.8157	−2.4687	1.5958	2.4547	12.1200	0.0023	52
Friday	−0.0009	0.0035	−0.9515	−2.8799	1.6590	2.5519	14.8058	0.0006	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.40: USD/JPY - Test for Departure from Normality

Year 2	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	0.0004	0.0034	0.1799	0.5445	0.1275	0.1961	0.3350	0.8458	52
Tuesday	0.0012	0.0033	−0.2926	−0.8934	0.0242	0.0375	0.7996	0.6705	53
Wednesday	0.0005	0.0043	0.6582	1.9920	1.0929	1.6811	6.7943	0.0335	52
Thursday	−0.0011	0.0044	−0.3357	−1.0159	−0.3875	−0.5961	1.3873	0.4997	52
Friday	−0.0003	0.0034	−0.0068	−0.0205	0.7257	1.1164	1.2467	0.5361	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.41: USD/JPY - Test for Departure from Normality

Year 3	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	0.0007	0.0040	0.5494	1.6779	1.8493	2.8697	11.0507	0.0040	53
Tuesday	0.0002	0.0049	−0.1776	−0.5374	0.1115	0.1715	0.3182	0.8529	52
Wednesday	−0.0011	0.0058	−0.8749	−2.6479	1.2089	1.8596	10.4695	0.0053	52
Thursday	0.0000	0.0043	0.3120	0.9444	0.3123	0.4804	1.1227	0.5704	52
Friday	−0.0009	0.0040	0.2088	0.6318	−0.5380	−0.8276	1.0841	0.5816	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.42: USD/JPY - Test for Departure from Normality

Year 4	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	0.0000	0.0074	0.5152	1.5592	0.8970	1.3798	4.3351	0.1145	52
Tuesday	0.0004	0.0073	−0.1292	−0.3910	−0.2822	−0.4340	0.3413	0.8431	52
Wednesday	0.0008	0.0066	0.6896	2.0870	0.8587	1.3208	6.1001	0.0474	52
Thursday	−0.0002	0.0084	−1.0099	−3.0565	2.4051	3.6996	23.0291	0.0000	52
Friday	−0.0010	0.0097	−1.5129	−4.5789	4.8681	7.4884	77.0422	0.0000	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.43: USD/JPY - Test for Departure from Normality

Year 5	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	−0.0002	0.0060	−2.4537	−7.4263	11.1796	17.1970	350.8858	0.0000	52
Tuesday	−0.0003	0.0043	−0.3493	−1.0572	2.9455	4.5309	21.6469	0.0000	52
Wednesday	−0.0011	0.0060	−1.0725	−3.2459	0.8439	1.2981	12.2209	0.0022	52
Thursday	−0.0001	0.0040	−0.1555	−0.4707	0.0650	0.0999	0.2316	0.8907	52
Friday	0.0006	0.0057	0.4864	1.4855	1.1201	1.7382	5.2278	0.0732	53

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.44: USD/JPY - Test for Departure from Normality

Year 6	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	0.0003	0.0041	0.5132	1.5532	2.8327	4.3574	21.3991	0.0000	52
Tuesday	−0.0001	0.0050	−0.5570	−1.6858	0.6063	0.9326	3.7117	0.1563	52
Wednesday	0.0008	0.0058	0.3102	0.9388	1.7879	2.7502	8.4450	0.0147	52
Thursday	0.0002	0.0041	−0.4226	−1.2907	1.0870	1.6868	4.5113	0.1048	53
Friday	0.0016	0.0062	1.7594	5.3248	5.8470	8.9941	109.2466	0.0000	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.45: USD/JPY - Test for Departure from Normality

Year 7	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	−0.0010	0.0067	0.0591	0.1790	1.1059	1.7011	2.9257	0.2316	52
Tuesday	0.0008	0.0067	−1.4732	−4.4586	6.2062	9.5467	111.0184	0.0000	52
Wednesday	0.0000	0.0064	−0.2292	−0.6999	0.3318	0.5149	0.7550	0.6856	53
Thursday	0.0013	0.0091	0.7426	2.2474	2.7659	4.2546	23.1522	0.0000	52
Friday	0.0005	0.0072	−0.0241	−0.0730	−0.7126	−1.0961	1.2067	0.5470	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.46: USD/JPY - Test for Departure from Normality

Year 8	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	−0.0004	0.0048	1.1033	3.3391	2.4864	3.8246	25.7772	0.0000	52
Tuesday	0.0002	0.0049	−0.0898	−0.2744	0.4389	0.6811	0.5392	0.7637	53
Wednesday	0.0016	0.0050	0.1727	0.5228	−0.7047	−1.0840	1.4484	0.4847	52
Thursday	0.0009	0.0060	0.7129	2.1577	1.5088	2.3209	10.0424	0.0066	52
Friday	0.0008	0.0054	0.1568	0.4747	−0.0594	−0.0914	0.2337	0.8897	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.47: USD/JPY - Test for Departure from Normality

Year 9	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	0.0001	0.0054	3.5821	10.9396	20.0292	31.0809	1085.6985	0.0000	53
Tuesday	0.0001	0.0056	−0.5396	−1.6331	1.9336	2.9743	11.5135	0.0032	52
Wednesday	−0.0004	0.0057	−3.4524	−10.4488	19.6665	30.2518	1024.3493	0.0000	52
Thursday	0.0003	0.0057	1.8675	5.6520	7.4748	11.4980	164.1490	0.0000	52
Friday	−0.0010	0.0062	0.7900	2.3910	2.6910	4.1393	22.8509	0.0000	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.48: USD/JPY - Test for Departure from Normality

Year 10	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	−0.0001	0.0055	0.0245	0.0741	0.5357	0.8240	0.6845	0.7102	52
Tuesday	−0.0011	0.0057	0.2980	0.9020	−0.4822	−0.7417	1.3637	0.5057	52
Wednesday	0.0014	0.0074	1.6908	5.1172	4.7540	7.3128	79.6628	0.0000	52
Thursday	−0.0016	0.0080	−1.6865	−5.1042	6.4534	9.9269	124.5961	0.0000	52
Friday	−0.0001	0.0055	0.0245	0.0741	0.5357	0.8240	0.6845	0.7102	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

E.5 EUR/JPY

E.5.1 Statistics Partitioned by Year

TABLE E.49: EUR/JPY - Test for Departure from Normality

Year	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Yr. 1	−0.0002	0.0036	−0.3317	−2.2004	1.3090	4.3576	23.8300	0.0000	261
Yr. 2	−0.0002	0.0047	−0.3071	−2.0370	1.1589	3.8579	19.0330	0.0000	261
Yr. 3	0.0004	0.0050	−0.3760	−2.4942	1.6211	5.3965	35.3435	0.0000	261
Yr. 4	−0.0002	0.0077	−2.0037	−13.2653	15.8087	52.5268	2935.0283	0.0000	260
Yr. 5	−0.0003	0.0059	0.3864	2.5632	1.6417	5.4651	36.4367	0.0000	261
Yr. 6	−0.0001	0.0054	−0.2721	−1.8048	2.9094	9.6853	97.0624	0.0000	261
Yr. 7	0.0004	0.0073	0.4051	2.6871	4.7772	15.9029	260.1239	0.0000	261
Yr. 8	0.0007	0.0076	0.5087	3.3741	0.7225	2.4052	17.1693	0.0002	261
Yr. 9	−0.0003	0.0085	−0.2077	−1.3776	1.8778	6.2510	40.9729	0.0000	261
Yr. 10	−0.0004	0.0094	−0.6110	−4.0453	3.2074	10.6569	129.9344	0.0000	260

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

E.5.2 Statistics Partitioned by Days of The Week

TABLE E.50: EUR/JPY - Test for Departure from Normality

Day of Week	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value[f]	Nbr. of Obs.
Monday	−0.0003	0.0061	−0.2567	−2.4012	2.6626	12.4765	161.4276	0.0000	522
Tuesday	0.0000	0.0064	−0.4245	−3.9704	1.7495	8.1979	82.9700	0.0000	522
Wednesday	0.0002	0.0065	−0.3555	−3.3256	4.2037	19.6982	399.0780	0.0000	522
Thursday	−0.0002	0.0076	−0.0385	−0.3596	5.6734	26.5594	705.5334	0.0000	521
Friday	−0.0002	0.0071	−0.8738	−8.1659	12.0113	56.2303	3228.5249	0.0000	521

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

E.5.3 Statistics Partitioned by Year and Days of The Week

TABLE E.51: EUR/JPY - Test for Departure from Normality

Year 1	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value[f]	Nbr. of Obs.
Monday	0.0006	0.0029	−0.4062	−1.2294	0.1858	0.2858	1.5932	0.4509	52
Tuesday	−0.0002	0.0033	0.7862	2.3794	0.4880	0.7506	6.2250	0.0445	52
Wednesday	0.0000	0.0031	−0.0306	−0.0935	2.9519	4.5808	20.9923	0.0000	53
Thursday	−0.0006	0.0042	−0.4788	−1.4492	0.7334	1.1281	3.3726	0.1852	52
Friday	−0.0008	0.0042	−0.4633	−1.4021	1.1141	1.7138	4.9032	0.0862	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.52: EUR/JPY - Test for Departure from Normality

Year 2	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	0.0010	0.0039	0.0948	0.2869	1.7801	2.7382	7.5800	0.0226	52
Tuesday	0.0000	0.0043	−0.3154	−0.9633	0.7197	1.1167	2.1751	0.3370	53
Wednesday	−0.0001	0.0053	−0.9656	−2.9224	2.3677	3.6421	21.8053	0.0000	52
Thursday	−0.0015	0.0053	−0.0192	−0.0581	0.6793	1.0449	1.0952	0.5783	52
Friday	−0.0005	0.0046	0.4201	1.2715	0.3644	0.5605	1.9308	0.3808	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.53: EUR/JPY - Test for Departure from Normality

Year 3	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	0.0005	0.0052	0.0858	0.2620	4.7061	7.3029	53.4003	0.0000	53
Tuesday	0.0009	0.0055	0.9203	2.7853	2.3570	3.6257	20.9036	0.0000	52
Wednesday	−0.0004	0.0047	−0.7103	−2.1497	0.7436	1.1438	5.9296	0.0516	52
Thursday	0.0006	0.0049	−0.1455	−0.4403	−0.1451	−0.2232	0.2437	0.8853	52
Friday	0.0003	0.0049	0.3190	0.9654	−0.6346	−0.9762	1.8849	0.3897	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.54: EUR/JPY - Test for Departure from Normality

Year 4	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	−0.0002	0.0074	0.4155	1.2574	1.2331	1.8968	5.1789	0.0751	52
Tuesday	0.0001	0.0065	0.2118	0.6412	0.4463	0.6865	0.8824	0.6433	52
Wednesday	0.0009	0.0048	0.4486	1.3578	0.5064	0.7789	2.4502	0.2937	52
Thursday	0.0002	0.0084	−0.5979	−1.8096	2.7407	4.2159	21.0485	0.0000	52
Friday	−0.0018	0.0103	−3.9740	−12.0274	22.1886	34.1315	1309.6147	0.0000	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.55: EUR/JPY - Test for Departure from Normality

Year 5	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	−0.0005	0.0044	−0.0401	−0.1213	−0.4050	−0.6230	0.4029	0.8175	52
Tuesday	−0.0011	0.0056	0.1020	0.3086	0.4016	0.6177	0.4768	0.7879	52
Wednesday	0.0002	0.0063	0.0268	0.0810	0.3075	0.4730	0.2303	0.8912	52
Thursday	0.0005	0.0071	0.3637	1.1008	1.6769	2.5795	7.8653	0.0196	52
Friday	−0.0005	0.0058	0.9756	2.9794	4.0474	6.2807	48.3246	0.0000	53

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.56: EUR/JPY - Test for Departure from Normality

Year 6	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	0.0005	0.0043	−0.4566	−1.3820	2.0090	3.0778	11.3829	0.0034	52
Tuesday	0.0001	0.0056	0.2589	0.7836	0.2826	0.4348	0.8031	0.6693	52
Wednesday	−0.0004	0.0048	−0.3129	−0.9470	0.9673	1.4880	3.1109	0.2111	52
Thursday	−0.0008	0.0065	−0.4478	−1.3676	3.2590	5.0573	27.4466	0.0000	53
Friday	0.0000	0.0058	−0.1992	−0.6028	5.0937	7.8354	61.7565	0.0000	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.57: EUR/JPY - Test for Departure from Normality

Year 7	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	−0.0008	0.0079	−0.7188	−2.1754	2.7040	4.1594	22.0333	0.0000	52
Tuesday	0.0012	0.0063	−0.7777	−2.3537	4.5125	6.9413	53.7217	0.0000	52
Wednesday	−0.0002	0.0052	0.1138	0.3476	−0.0135	−0.0209	0.1213	0.9412	53
Thursday	0.0018	0.0091	1.5674	4.7436	6.2932	9.6805	116.2143	0.0000	52
Friday	0.0002	0.0071	0.1293	0.3913	−0.0050	−0.0077	0.1532	0.9263	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.58: EUR/JPY - Test for Departure from Normality

Year 8	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	−0.0007	0.0069	0.3417	1.0342	0.7549	1.1612	2.4179	0.2985	52
Tuesday	0.0001	0.0063	−0.1456	−0.4448	0.8242	1.2790	1.8338	0.3998	53
Wednesday	0.0009	0.0070	−0.0522	−0.1579	−0.0794	−0.1222	0.0398	0.9803	52
Thursday	0.0011	0.0092	0.6996	2.1174	0.5347	0.8226	5.1601	0.0758	52
Friday	0.0019	0.0084	0.7158	2.1662	0.2988	0.4596	4.9038	0.0861	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.59: EUR/JPY - Test for Departure from Normality

Year 9	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	−0.0013	0.0084	−0.4255	−1.2994	0.9044	1.4035	3.6582	0.1606	53
Tuesday	0.0014	0.0075	−0.4789	−1.4494	0.8863	1.3633	3.9593	0.1381	52
Wednesday	−0.0015	0.0098	−0.6562	−1.9860	2.5690	3.9518	19.5609	0.0001	52
Thursday	0.0013	0.0080	−0.3568	−1.0800	0.9681	1.4891	3.3839	0.1842	52
Friday	−0.0013	0.0083	1.3932	4.2166	4.4403	6.8302	64.4320	0.0000	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.60: EUR/JPY - Test for Departure from Normality

Year 10	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	0.0004	0.0084	−0.4671	−1.4136	0.8754	1.3465	3.8116	0.1487	52
Tuesday	−0.0026	0.0099	−0.6324	−1.9140	−0.5610	−0.8629	4.4079	0.1104	52
Wednesday	0.0022	0.0101	−0.3829	−1.1589	2.4978	3.8422	16.1058	0.0003	52
Thursday	−0.0002	0.0108	−1.3424	−4.0629	7.3897	11.3671	145.7176	0.0000	52
Friday	0.0004	0.0084	−0.4671	−1.4136	0.8754	1.3465	3.8116	0.1487	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

E.6 GBP/JPY

E.6.1 Statistics Partitioned by Year

TABLE E.61: GBP/JPY - Test for Departure from Normality

Year	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Yr. 1	0.0000	0.0056	0.6705	4.4477	2.3619	7.8625	81.6003	0.0000	261
Yr. 2	−0.0001	0.0057	−0.0350	−0.2322	0.7362	2.4508	6.0602	0.0000	261
Yr. 3	0.0002	0.0059	−2.0222	−13.4135	1.1346	3.7770	194.1872	0.0000	261
Yr. 4	−0.0006	0.0124	−3.6298	−24.0314	35.4406	117.7570	14 444.2146	0.0000	260
Yr. 5	−0.0005	0.0061	−0.4056	−2.6902	1.7703	5.8932	41.9669	0.0000	261
Yr. 6	0.0003	0.0052	0.5071	3.3636	3.1143	10.3673	118.7951	0.0000	261
Yr. 7	0.0006	0.0073	0.3188	2.1146	4.5598	15.1792	234.8793	0.0000	261
Yr. 8	0.0005	0.0064	0.1410	0.9355	0.7190	2.3936	6.6046	0.0000	261
Yr. 9	−0.0003	0.0073	0.0531	0.3524	2.8811	9.5910	92.1123	0.0000	261
Yr. 10	−0.0002	0.0091	−0.8479	−5.6135	7.3134	24.2998	621.9936	0.0000	260

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

E.6.2 Statistics Partitioned by Days of The Week

TABLE E.62: GBP/JPY - Test for Departure from Normality

Day of Week	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	−0.0003	0.0064	−0.3326	−3.1113	3.6743	17.2173	306.1168	0.0000	522
Tuesday	0.0000	0.0071	−0.0492	−0.4606	2.2247	10.4245	108.8816	0.0000	522
Wednesday	0.0007	0.0068	0.0271	0.1940	3.0070	14.0906	198.5829	0.0000	522
Thursday	0.0001	0.0077	−0.4829	−4.5129	8.7567	40.9938	1700.8602	0.0000	521
Friday	−0.0005	0.0088	−5.0314	−47.0195	72.4487	339.1640	117 243.0587	0.0000	521

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

E.6.3 Statistics Partitioned by Year and Days of The Week

TABLE E.63: GBP/JPY - Test for Departure from Normality

Year 1	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	−0.0006	0.0051	−0.0528	−0.1597	0.4004	0.6159	0.4049	0.8167	52
Tuesday	0.0003	0.0057	0.3358	1.0163	1.6330	2.5120	7.3429	0.0254	52
Wednesday	0.0010	0.0054	1.0187	3.1111	2.7787	4.3120	28.2721	0.0000	53
Thursday	−0.0005	0.0060	1.1040	3.3412	5.0379	7.7495	71.2180	0.0000	52
Friday	−0.0004	0.0059	0.8453	2.5582	2.0580	3.1657	16.5660	0.0003	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.64: GBP/JPY - Test for Departure from Normality

Year 2	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	0.0006	0.0044	0.2109	0.6382	0.4195	0.6452	0.8236	0.6625	52
Tuesday	0.0007	0.0052	−0.1591	−0.4860	−0.6302	−0.9779	1.1920	0.5509	53
Wednesday	0.0002	0.0064	−0.7136	−2.1598	1.9183	2.9507	13.3716	0.0012	52
Thursday	−0.0017	0.0063	0.5868	1.7760	0.9128	1.4040	5.1255	0.0771	52
Friday	−0.0004	0.0056	0.4656	1.4091	0.7418	1.1411	3.2877	0.1932	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.65: GBP/JPY - Test for Departure from Normality

Year 3	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	0.0002	0.0053	−1.2576	−3.8406	4.1477	6.4363	26.1759	0.0000	53
Tuesday	0.0008	0.0061	0.8566	2.5926	0.7212	1.1094	7.9524	0.0188	52
Wednesday	0.0002	0.0054	−0.5255	−1.5904	1.0078	1.5503	4.9327	0.0849	52
Thursday	0.0009	0.0063	−0.1609	−0.4871	0.2204	0.3390	0.3522	0.8385	52
Friday	−0.0009	0.0064	0.5078	1.5370	1.5130	2.3274	7.7791	0.0205	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.66: GBP/JPY - Test for Departure from Normality

Year 4	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value[f]	Nbr. of Obs.
Monday	−0.0003	0.0102	−0.5047	−1.5275	2.1361	3.2858	13.1300	0.0014	52
Tuesday	0.0003	0.0113	0.1947	0.5893	1.4877	2.2884	5.5841	0.0613	52
Wednesday	0.0014	0.0079	0.2624	0.7943	−0.2356	−0.3624	0.7622	0.6831	52
Thursday	0.0002	0.0101	0.1824	0.5521	1.2148	1.8687	3.7967	0.1498	52
Friday	−0.0044	0.0193	−4.6643	−14.1165	28.5778	43.9595	2131.7152	0.0000	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.67: GBP/JPY - Test for Departure from Normality

Year 5	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value[f]	Nbr. of Obs.
Monday	−0.0002	0.0063	−1.1590	−3.5077	3.9658	6.1003	49.5176	0.0000	52
Tuesday	−0.0014	0.0061	−0.2966	−0.8976	0.3385	0.5207	1.0768	0.5837	52
Wednesday	0.0003	0.0060	−0.8508	−2.5748	2.3221	3.5207	19.3883	0.0001	52
Thursday	−0.0006	0.0051	−0.1371	−0.4150	−0.3900	−0.5999	0.5321	0.7664	52
Friday	−0.0005	0.0070	0.1941	0.5927	1.9269	2.9901	9.2919	0.0096	53

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.68: GBP/JPY - Test for Departure from Normality

Year 6	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value[f]	Nbr. of Obs.
Monday	0.0003	0.0048	−0.2302	−0.6968	1.9550	3.0072	9.5288	0.0085	52
Tuesday	0.0001	0.0052	−0.1460	−0.4420	1.9167	2.9483	8.8880	0.0117	52
Wednesday	0.0007	0.0057	−0.1155	−0.3495	−0.0217	−0.0334	0.1233	0.9402	52
Thursday	0.0001	0.0049	0.1962	0.5993	0.5978	0.9276	1.2197	0.5434	53
Friday	0.0002	0.0056	2.2903	6.9316	10.3068	15.8543	299.4057	0.0000	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.69: GBP/JPY - Test for Departure from Normality

Year 7	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	−0.0006	0.0075	−0.9616	−2.9102	2.2179	3.4116	20.1081	0.0000	52
Tuesday	0.0004	0.0066	−0.7272	−2.2008	3.1835	4.8971	28.8246	0.0000	52
Wednesday	−0.0009	0.0061	0.6681	2.0405	2.7350	4.2442	22.1767	0.0000	53
Thursday	0.0028	0.0085	1.6577	5.0170	7.4732	11.4955	157.3169	0.0000	52
Friday	−0.0003	0.0072	−0.0955	−0.2890	0.6524	1.0036	1.0907	0.5796	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.70: GBP/JPY - Test for Departure from Normality

Year 8	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	−0.0007	0.0054	0.2911	0.8809	0.0676	0.1039	0.7868	0.6748	52
Tuesday	0.0000	0.0055	−0.8423	−2.5722	0.9576	1.4859	8.8243	0.0121	53
Wednesday	0.0009	0.0064	−0.1321	−0.3997	−0.4031	−0.6200	0.5442	0.7618	52
Thursday	0.0011	0.0079	−0.1229	−0.3721	0.4838	0.7442	0.6923	0.7074	52
Friday	0.0012	0.0065	1.0107	3.0589	1.4974	2.3033	14.6625	0.0007	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.71: GBP/JPY - Test for Departure from Normality

Year 9	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	−0.0007	0.0073	0.8496	2.5948	3.6249	5.6251	38.3746	0.0000	53
Tuesday	0.0004	0.0077	−0.3988	−1.2070	0.7051	1.0846	2.6331	0.2681	52
Wednesday	−0.0013	0.0088	−0.7552	−2.2855	3.2175	4.9493	29.7197	0.0000	52
Thursday	0.0001	0.0061	0.4015	1.2151	0.7921	1.2185	2.9612	0.2275	52
Friday	0.0002	0.0066	1.6122	4.8794	5.0069	7.7018	83.1267	0.0000	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

TABLE E.72: GBP/JPY - Test for Departure from Normality

Year 10	Mean	Std. Dev.	Skewness S_k^a	Z_{Sk}^b	Kurtosis K_u^c	Z_{Ku}^d	DP^e	p-value^f	Nbr. of Obs.
Monday	−0.0001	0.0081	−0.2971	−0.8992	1.1515	1.7714	3.9462	0.1390	52
Tuesday	−0.0017	0.0092	0.0780	0.2360	−0.3557	−0.5471	0.3550	0.8374	52
Wednesday	0.0023	0.0087	0.9265	2.8041	3.8660	5.9468	43.2277	0.0000	52
Thursday	−0.0009	0.0120	−2.2896	−6.9295	10.3004	15.8445	299.0651	0.0000	52
Friday	−0.0001	0.0081	−0.2971	−0.8992	1.1515	1.7714	3.9462	0.1390	52

[a] Skewness statistic is the third central moment divided by the three-halves power of the second central moment multiplied by the sample skewness coefficient. [b] The skewness test statistic is the sample skewness divided by the standard error of skewness. The 5% confidence interval for skewness is sample skewness ± 1.96 times the standard error of skewness. [c] The kurtosis coefficient is the fourth central moment divided by the square of the second central moment multiplied by the sample kurtosis coefficient. [d] The kurtosis test statistic is the sample kurtosis divided by the standard error of kurtosis. The 5% confidence interval for kurtosis is sample kurtosis ± 1.96 times the standard error of kurtosis. [e] The D'Agostino-Pearson test statistic with χ^2 2df. [f] The p-value at the 5% level of significance.

Appendix F

Statistical Analysis of Skewness and Kurtosis

F.1 Statical Analysis of Skewness and Kurtosis

If the absolute value of the skewness for the data is more than twice the standard error of skewness, (SES), this indicates that the data is not likely symmetric, and therefore not normal. Similarly if the absolute value of the kurtosis for the data is more than twice the standard error of kurtosis, (SEK), this is also an indication that the data is unlikely to be normally distributed.

The moment coefficient of the sample skewness (Sk) of a data set is given by

$$\begin{aligned} Sk &= \frac{\sqrt{n(n-1)}}{n-2} \cdot m_3/m_2^{\frac{3}{2}}\,, \\ \text{where} \quad m_3 &= \frac{\sum(x-\bar{x})^3}{n} \qquad \text{and} \qquad m_2 = \frac{\sum(x-\bar{x})^2}{n}\,, \end{aligned} \tag{F.1}$$

where $\bar{x}$ is the sample mean and n is the sample size, m_3 is the third moment of the data set and m_2 is the variance. If the sample skewness is positive, the data set is positively skewed to the right: the right tail of the distribution is longer than the left tail. Conversely for a negative skewness the data is skewed to the left with that tail longer. If the skewness $= 0$, the data is perfectly symmetrical. Although a skewness of exactly zero is unlikely to occur, raising the question of how to interpret skewness? Bulmer (1979) suggests the following rule:

i. If $-1 \leq Sk \geq +1$, the distribution is *highly skewed*.

ii. If $-1 \geq Sk \geq -\frac{1}{2}$ or between $+\frac{1}{2} \leq Sk \leq +1$, the distribution is *moderately skewed*.

iii. If skewness is between $-\frac{1}{2} \geq Sk \leq +\frac{1}{2}$, the distribution is *approximately symmetric*.

The measure of skewness is an interpretation of the sample data. The sample skewness does not necessarily apply to the whole population. The question arises, from the sample skewness, can anything be concluded about the population from the sample? The data set is just one sample of data drawn from a population. The sample could be skewed even though the population is symmetric. But if the sample is skewed too much for random change to be an explanation, then the conclusion drawn is that there is skewness in the population. To address the question of *too much for random chance to be the explanations?* consider the Cramer (2002) standard error of skewness (SES), where SES is given by

$$SES = \sqrt{\frac{6n(n-1)}{(n-2)(n+1)(n+3)}}\,, \tag{F.2}$$

where n is the sample size. Dividing the sample skewness (Sk) by the standard error of skewness (SES) gives a measure of how many standard errors separate the sample skewness from zero where the test statistic Z_{Sk} has a standard normal distribution $\sim N(0,1)$ and given by

$$Z_{Sk} = \frac{Sk}{SES}\,. \tag{F.3}$$

The critical value of Z_{Sk} is approximately 1.96 for a two-tailed test of skewness $\neq 0$ at the 5% level of significance.

i. If $Z_{Sk} < -1.96$, the population is very likely skewed negatively, although there is no indication by how much.

ii. If $-1.96 \geq Z_{Sk} \leq +1.96$, you cannot reach any conclusions about the skewness of the population: it might be symmetric, or it might be skewed in either direction.

iii. If $Z_{Sk} > 1.96$, the population is very likely skewed positively with no indication by how much.

A distinction between the test statistic (Z_{Sk}) and the amount of skewness (Sk) must be made. The amount of skewness is an indication of how highly skewed the sample is. The test statistic tells us whether the whole population is probably skewed but not by how much. The larger the test statistic the greater the probability of the population being skewed.

If a distribution is symmetric, the next question is about the central peak: is it high and sharp or short and broad? The height and sharpness of the peak relative to the rest of the data is measured by the kurtosis. Higher values of kurtosis indicate a higher, sharper peak: more of the variability is due to a few extreme differences from the mean; lower values indicate a lower, less distinct peak due to more modest differences from the mean.

Balanda and MacGillivray (1988) noted that increasing kurtosis is associated with the movement of probability mass from the shoulders of a distribution into its centre and tails. The referenced standard is a normal distribution, which has a kurtosis of 3. Excess kurtosis is simply Kurtosis-3 and classified as:

i. Mesokurtic: a normal distribution with kurtosis exactly 3.

ii. Platykurtic: kurtosis < 3. Compared to a normal distribution its central peak is lower and broader, and its tails are shorter and thinner.

iii. Leptokurtic kurtosis > 3. Compared to a normal distribution its central peak is higher and sharper and its tails are longer and fatter.

The smallest possible kurtosis is 1 (excess kurtosis minus 2), and the largest is ∞.

The Joanes and Gill (1998) moment coefficient of kurtosis (Ku) of a data set is given by

$$Ku = \frac{(n-1)}{(n-2)(n-3)} \cdot \left[(n+1)\, m_4/m_2^2\right], \qquad \text{where} \quad m_4 = \frac{\sum(x-\bar{x})^4}{n} \quad \text{and} \quad m_2 = \frac{\sum(x-\bar{x})^2}{n}, \tag{F.4}$$

where $\bar{x}$ is the sample mean and n is the sample size, m_4 is the fourth moment of the data set and m_2 is the variance.

The sample kurtosis is an interpretation of the sample data. The sample kurtosis does not necessarily apply to the whole population. To determine what the sample kurtosis tells us about the population kurtosis the Cramer (2002) standard error of kurtosis (SEK) is applied where the SEK is given by

$$SEK = 2 \cdot (SES) \sqrt{\frac{(n^2-1)}{(n-3)(n+5)}}, \tag{F.5}$$

where n is the sample size. Dividing the sample kurtosis (Ku) by the standard error of kurtosis (SEK) gives a measure of how many standard errors separate the sample kurtosis from zero where the test statistic Z_{Ku} has a standard normal distribution $\sim N(0,1)$ and given by

$$Z_{Ku} = \frac{Ku}{SEK}. \tag{F.6}$$

The critical value of Z_{Ku} is approximately 1.96 for a two-tailed test of kurtosis $\neq 0$ at the 5% level of significance.

i. If $Z_{Ku} < -1.96$, the population very likely has negative excess kurtosis: platykurtic, although there is no indication by how much.

ii. If $-1.96 \geq Z_{Ku} \leq +1.96$, you cannot reach any conclusions about the kurtosis of the population: it might be mesokurtic, platykurtic or leptokurtic.

iii. If $Z_{Ku} > +1.96$, the population very likely has positive excess kurtosis: leptokurtic, although there is no indication by how much.

Appendix G

Black-Scholes Methodology and Assumptions

G.1 Methodology and Assumptions

A European option is an option that can only be exercised on a specified future date. The price paid for the asset is the exercise or strike price and the last day on which the option may be exercised is the expiration or maturity date.

The inherent characteristics of a call option conform to the following qualities:

i. The higher the price of the stock, the greater the value of the option.

ii. When the stock price is much greater than the strike price the option is almost certain to be exercised.

iii. The value of the option will be equal to the price of the stock minus the price of a pure discount bond that matures on the same date as the option, with a face value equal to the strike price.

iv. Conversely if the stock price is much less than the strike price the option will expire without being exercised so its value will be zero.

v. The option value is constrained by the boundary conditions of the option.

vi. The maximum value of the option cannot be more than that of the stock.

vii. The minimum value cannot be less than the stock price minus the exercise price.

viii. If the expiration date is far into the future the value of the option will approximately be equal to the price of the stock.

ix. If the expiration date is very near the value of the option will be the stock price minus the exercise price or zero if the stock price is less than the strike price.

x. The value of an option declines as its maturity date approaches, if the value of the stock does not change.

xi. The option is more volatile than the stock. A percentage change in the stock price, holding maturity constant will result in a larger percentage change in the option value.

xii. The relative volatility of the option is not constant: it depends on the stock price and maturity.

G.1.1 The Black-Scholes Equation

The price of any derivative is a function of the stochastic variables underlying the derivative and time. Thus the price of a stock option is a function of the underlying stock price and time. The main principles used to formulate the Black-Scholes differential equation are:

i. No arbitrage transactions: This proposes that in a perfectly competitive, liquid market, no opportunities exist to earn a risk-free profit.

ii. The creation of a riskless portfolio: At any time t, the proportion of each investment in the portfolio must be set so that the net effect due to a small change in the price of the underlying asset on the value of the portfolio is zero. A risk-free investment portfolio must earn the risk-free rate of return.

If the conditions in the Black and Scholes (1973) model are satisfied and the percentage return of the portfolio is equal to the risk-free interest rate then the value of a portfolio at the end of a short period of time is known with certainty and is riskless only for that instantaneously short period of time.

Methodology and Assumptions

The Black and Scholes (1973) no arbitrage principle depends on the assumptions that it is possible to buy or sell any finite quantity of the underlying asset at any time: perfect liquidity, and that trading is continuous in time. Further, the full use of the proceeds from short selling securities is permitted.

Black and Scholes (1973) make the following set of assumptions in deriving the differential equation:

i. The stock price follows a geometric Brownian motion with constant drift μ and constant volatility σ. Thus the distribution of possible stock prices at the end of any finite interval is log-normal. The variance rate of the return on the stock is constant.

ii. There are no penalties for short selling. A seller who does not own a security will simply accept the price of the security from the buyer, and will agree to settle with the buyer on some future date by paying him an amount equal to the price of the security on that date.

iii. There are no transaction costs or taxes in buying or selling the stock or option.

iv. It is possible to borrow any fraction of the price of a security, to buy it or to hold it, at the risk-free interest rate. That is all securities are perfectly divisible.

v. The stock pays no dividends or other distributions.

vi. There are no riskless arbitrage opportunities.

vii. Security trading is continuous.

viii. The risk-free interest rate is known and is constant through time, for all maturities.

ix. The option is "European," that is, it can only be exercised at maturity.

Given that the market direction cannot be predicted, the stock price is represented by a log-normal random walk in continuous-time with a variance rate proportional to the square of the stock price. Thus the distribution of possible stock prices at the end of any finite interval is a log-normal geometric Brownian motion with constant drift μ and volatility σ: the expected return and the variance of stock prices are constant.

G.1.2 A Derivation

A European call option with strike price K and maturity T gives the buyer the right, but not the obligation, to purchase the underlying stock at price K on the maturity or expiry date T. Suppose the price of a contingent claim is $f(S,t)$, which derives its value from the performance of a tradeable equity security with asset price S at time t. At maturity the call option is worth the difference between the stock price S and the strike price K if the stock price is greater than the strike price $S > K$. If the strike price is greater than the stock price $K > S$ then the option is worthless. That is the price of the call option satisfies the condition

$$f(S,T) = max(S-K,0)\,. \tag{G.1}$$

This is the fundamental boundary condition used to obtain a closed-form solution for the option price as a function of the stock price and time. At maturity, the value of a call option is the spot price minus the contract price, if that is positive, or zero: Black (1976).

The Price Process and Diffusion

A fundamental principle of the Black and Scholes (1973) formula is the creation of a riskless portfolio consisting of a certain proportion of shares and options satisfying the boundary conditions. In creating this portfolio Black and Scholes (1973) assumed that the stochastic process for the stock price followed a geometric Brownian motion

$$\frac{dS}{S} = \mu\,dt + \sigma\,dz\,, \tag{G.2}$$

where S is the stock price and dS/S follows a geometric Brownian motion with expected rate of return μ and variance σ^2.

Using this assumption and letting $f(S,t)$ denote the price of the option derived from the stock price S at time t; applying Itô's lemma the diffusion becomes

$$df = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right) dt + \frac{\partial f}{\partial S}\sigma S\, dz\,. \tag{G.3}$$

The discrete version of Equations (G.2) and (G.3) are

$$\Delta S = \mu S\,\Delta t + \sigma S\,\Delta z \tag{G.4}$$

and

$$\Delta f = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right) \Delta t + \frac{\partial f}{\partial S}\sigma S \Delta z\,, \tag{G.5}$$

where ΔS and Δf are the changes in S and f over the small time interval Δt and Δz is a Wiener process.

To eliminate the Wiener process, Δz, the portfolio would consist of one short derivative contract and long an amount of $\frac{\partial f}{\partial S}$ shares of the underlying stock. Define Π as the value of this portfolio, then

$$\Pi = -f + \frac{\partial f}{\partial S}\,S\,. \tag{G.6}$$

The change $\Delta\Pi$ in the value of the portfolio over the interval Δt is given by

$$\Delta\Pi = -\Delta f + \frac{\partial f}{\partial S}\,\Delta S\,. \tag{G.7}$$

Under the assumption that $\frac{\partial f}{\partial S}$ is constant over the time period Δt and substituting in Equations (G.4) and (G.5) and cancelling out the Wiener term yields

$$\Delta\Pi = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right) \Delta t = r\Pi\,\Delta t\,, \tag{G.8}$$

where Equation (G.8) does not contain a stochastic term and the portfolio is riskless during the time interval Δt, thus the return must be equal to the risk-free rate of interest, r. Substituting for the value of the portfolio Π: Equation (G.6), gives

$$\Delta\Pi = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right) \Delta t = r\left(-f + \frac{\partial f}{\partial S}S\right) \Delta t\,. \tag{G.9}$$

Rearranging yields the Black-Scholes differential equation

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf\,. \tag{G.10}$$

Black and Scholes (1973) noted that there is only one formula $f(S,t)$ that satisfies Equation (G.10) subject to the boundary condition $f(S,t) = max(S-K,0)$ and this formula is the option valuation formula. The Black and Scholes (1973) differential equation must be satisfied by the price of any derivative dependent on a non-dividend paying stock: J. C. Hull (2012). To obtain the closed-form solution of Equation (G.10), Malone (2002) makes the following substitutions

$$\begin{aligned} & f(S,t) = e^{r(t-T)} g\left[A(S,t), B(S,t)\right]\,, \\ \text{where} \quad & A(S,t) = \left(\frac{2}{\sigma^2}\right)\left(r - \frac{1}{2}\sigma^2\right)\left[\ln\left(\frac{S}{K}\right) - \left(r - \frac{1}{2}\sigma^2\right)(t-T)\right] \\ \text{and} \quad & B(S,t) = -\left(\frac{2}{\sigma^2}\right)\left(r - \frac{1}{2}\sigma^2\right)^2 (t-T)\,, \end{aligned} \tag{G.11}$$

whereby Equation (G.10) simplifies to

$$\frac{\partial g}{\partial b} = \frac{\partial^2 g}{\partial a^2}\,, \tag{G.12}$$

and the boundary condition becomes

$$g(a,0) = \begin{cases} K\left[e^{a\left(\frac{1}{2}\sigma^2\right)/\left(r-\frac{1}{2}\sigma^2\right)} - 1\right] & \text{if} \quad a \geq 0 \\ 0, & \text{if} \quad a < 0\,, \end{cases} \tag{G.13}$$

substituting the resulting function $g(a, b)$ into Equation (G.12) and simplifying gives

$$
\begin{aligned}
f_c(S,t) &= SN(d_1) - Ke^{-rT}N(d_2)\,, \\
f_p(S,t) &= Ke^{-rT}N(-d_2) - SN(-d_1)\,, \\
\text{where} \quad d_1 &= \frac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \\
\text{and} \quad d_2 &= \frac{\ln(S/K) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\,,
\end{aligned}
\tag{G.14}
$$

where $N(\cdot)$ is the cumulative probability distribution function for a normally distributed variable with a mean of zero and a standard deviation of 1. S is the stock price at time $t = 0$, K is the strike price, r is the continuously compounded risk-free interest rate, σ is the stock price volatility and T is the time to maturity of the option.

Appendix H

Python Code

```
import xlsxwriter
import math
import numpy as np
import pandas as pd
#==================================
# Set up rounding half up function
#==================================
def round_half_up(n, decimals=0):
    multiplier = 10 ** decimals
    return math.floor(n*multiplier + 0.5) / multiplier

def round_half_away_from_zero(n, decimals=0):
    rounded_abs = round_half_up(abs(n), decimals)
    return math.copysign(rounded_abs, n)
#=========================================================
# Import data from generic workbook.xlsx file into pandas
#=========================================================
series_excel_data_ccy1ccy2=pd.read_excel('workbook.xlsx', sheet_name='ccy1ccy2')
#===============================================================================
# Slice the data, df_name.column_header_index.iloc[row index]: Isolate the 4th digit
#===============================================================================
#Iterate setting the value to each cell the final digit integer of that cell
EPObsz0, EPObs1, EPObs2, EPObs3, EPObs4, EPObs5, EPObs6, EPObs7, EPObs8, EPObs9
=0,0,0,0,0,0,0,0,0,0
for index in series_excel_data_ccy1ccy2.index:
    series_excel_data_ccy1ccy2.Mid.iloc[index]=int((series_excel_data_ccy1ccy2.
    Mid.iloc[index]+0.000055)*10000)-int((series_excel_data_ccy1ccy2.Mid.iloc[index]
    +0.000055)*1000)*10
    if  series_excel_data_ccy1ccy2.Mid.iloc[index]== 0:
        EPObszero+=1
    elif series_excel_data_ccy1ccy2.Mid.iloc[index]== 1:
        EPObs1+=1
    elif series_excel_data_ccy1ccy2.Mid.iloc[index]== 2:
        EPObs2+=1
    elif series_excel_data_ccy1ccy2.Mid.iloc[index]== 3:
        EPObs3+=1
    elif series_excel_data_ccy1ccy2.Mid.iloc[index]== 4:
        EPObs4+=1
    elif series_excel_data_ccy1ccy2.Mid.iloc[index]== 5:
        EPObs5+=1
```

```
    elif series_excel_data_ccy1ccy2.Mid.iloc[index]== 6:
        EPObs6+=1
    elif series_excel_data_ccy1ccy2.Mid.iloc[index]== 7:
        EPObs7+=1
    elif series_excel_data_ccy1ccy2.Mid.iloc[index]== 8:
        EPObs8+=1
    elif series_excel_data_ccy1ccy2.Mid.iloc[index]== 9:
        EPObs9+=1
EPTotalObs=EPObs0+EPObs1+EPObs2+EPObs3+EPObs4+EPObs5+EPObs6+EPObs7+EPObs8+EPObs9
#===================================================================
# To count the number of 0 to 9 digits in the sample
#===================================================================
EPdigit_count=[EPObs0, EPObs1, EPObs2, EPObs3, EPObs4, EPObs5, EPObs6, EPObs7,
EPObs8, EPObs9]
EPdigit_percent=[(EPObs0/EPTotalObs)*100, (EPObs1/EPTotalObs)*100,
(EPObs2/EPTotalObs)*100,
(EPObs3/EPTotalObs)*100, (EPObs4/EPTotalObs)*100, (EPObs5/EPTotalObs)*100,
(EPObs6/EPTotalObs)*100, (EPObs7/EPTotalObs)*100, (EPObs8/EPTotalObs)*100,
(EPObs9/EPTotalObs)*100]
EPdigit_decimal=[(EPObs0/EPTotalObs), (EPObs1/EPTotalObs), (EPObs2/EPTotalObs),
(EPObs3/EPTotalObs), (EPObs4/EPTotalObs), (EPObs5/EPTotalObs), (EPObs6/EPTotalObs),
(EPObs7/EPTotalObs), (EPObs8/EPTotalObs), (EPObs9/EPTotalObs)]
EPfinal_digit=[0,1,2,3,4,5,6,7,8,9]
#==========================
# Creating the Series
#==========================
EPdigit_count_series=pd.Series(EPdigit_count, EPfinal_digit)
EPdigit_percent_series=pd.Series(EPdigit_percent, EPfinal_digit)
EPdigit_decimal_series=pd.Series(EPdigit_decimal, EPfinal_digit)
#==================== Test values print out ========================
#===================================
# Chi Squared sum((o-e)^2/e)
#===================================
EPExpected=EPTotalObs/10
EPChiSquared=((EPObs0-EPExpected)**2+(EPObs1-EPExpected)**2+(EPObs2-EPExpected)**2+
(EPObs3-EPExpected)**2+(EPObs4-EPExpected)**2+(EPObs5-EPExpected)**2+(
EPObs6-EPExpected)**2+(EPObs7-EPExpected)**2+(EPObs8-EPExpected)**2+
(EPObs9-EPExpected)**2)/EPExpected
print('EP_ChiSquared=',round(EPChiSquared,3))
#=========================================================================
# Standardised Range (Hi%-Lo%)/if No clustering i.e. 10%
#=========================================================================
#==================
# Standised Range
#==================
EPSR=(EPdigit_percent_series.max()-EPdigit_percent_series.min())/10
print('EP_Standardised_Rage=', round(EPSR,4))
#============================
# Attraction Hypothesis
#============================
#====== Taking average pairs =====================
EPavg=[EPObs3,EPObs7]
EPavg37 = np.mean(EPavg)
EPavg=[EPObs2,EPObs8]
EPavg28 = np.mean(EPavg)
EPavg=[EPObs4,EPObs6]
EPavg46 = np.mean(EPavg)
EPavg28lessavg46=EPavg28-EPavg46
EPavg=[EPObs1,EPObs9]
```

```
EPavg19 = np.mean(EPavg)
#========= Take differences between groups ===========
EPavg37lessavg28=EPavg37-EPavg28
EPavg46lessavg19=EPavg46-EPavg19
#==============================
EPGroup=[abs((EPObs3-EPObs7)),abs((EPObs2-EPObs8)),abs((EPObs4-EPObs6)),
abs((EPObs1-EPObs9))]
EPGroupavg = np.mean(EPGroup)
EPAttraction=min(EPavg37lessavg28,EPavg28lessavg46,EPavg46lessavg19)/EPGroupavg
print('EP Attraction Test Statistic =',round(EPAttraction,4))
#======================
# Resolution Hypothesis
#======================
#=============== Take average per group =====================
EPavg=[EPObs2, EPObs3, EPObs7, EPObs8]
EPavg2378 = np.mean(EPavg)
EPavg=[EPObs1,EPObs4,EPObs6,EPObs9]
EPavg1469 = np.mean(EPavg)
#==== Take the difference between the average groups ==========
EPavg2378lessavg1469=EPavg2378-EPavg1469
#=============Take Max and Min values per group ==================
EPmax2378=max(EPObs2, EPObs3, EPObs7, EPObs8)
EPmin2378=min(EPObs2, EPObs3, EPObs7, EPObs8)
EPmax1469=max(EPObs1,EPObs4,EPObs6,EPObs9)
EPmin1469=min(EPObs1,EPObs4,EPObs6,EPObs9)
#====== Take difference between Max and Min values ==========
EPmax2378lessmin2378=EPmax2378-EPmin2378
EPmax1469lessmin1469=EPmax1469-EPmin1469
#=====Take Max of max-min pairs ==============
EPMaxdenom=max(EPmax2378lessmin2378, EPmax1469lessmin1469)
EPResolutiontion=EPavg2378lessavg1469/EPMaxdenom
print('EP Resolution Test Statistic =', round(EPResolutiontion,4))

#==========================================================================
# SAVE Results to_excel xlsx usinging .xlsxWriter
#============================================================================
# The most versatile and preferred method using xlsxWriter NB import first
#============================================================================
workbook = xlsxwriter.Workbook('Collusion Tests Results To Be Deleted.xlsx')
worksheetTables = workbook.add_worksheet('Tables')
worksheetGraphs = workbook.add_worksheet('Graphs')
bold = workbook.add_format({'bold': True})
bold_and_italic_bottomborder = workbook.add_format({'bold': True, 'italic': True,'bottom':1})
# Widen the first column to make the text clearer.
worksheetTables.set_column('A:A', 20)
# Write sample text abd numbers.
worksheetTables.write('A1', 'ccy1ccy2', bold_and_italic_bottomborder)
worksheetTables.write('A2', 'Chi Squared', bold)
worksheetTables.write('A3', 'Standard Range', bold)
worksheetTables.write('A4', 'Attraction', bold)
worksheetTables.write('A5', 'Resoltion', bold)
# Write some numbers, with row/column notation.
worksheetTables.write(1, 1, EPChiSquared)
worksheetTables.write(2, 1, EPSR)
worksheetTables.write(3, 1, round(EPAttraction,4))
worksheetTables.write(4, 1, round(EPResolutiontion,4))
#==============
# Plot SR Graphs
percent=workbook.add_format({'num_format':'0.00%'})
```

```
#worksheetgraph.write(1, 0, 3.1415926, percent)
#==========================================================================
# Add headings and data ccy1ccy2 graphs
#==========================================================================
worksheetGraphs.set_column('A:A', 10)
headings = ['Final_Digits', 'EURGBP_SR']
#data = [[0,1,2,3,4,5,6,7,8,9], [(PDObs0/PDTotalObs)*100, (PDObs1/PDTotalObs)*100,
(PDObs2/PDTotalObs)*100, (PDObs3/PDTotalObs)*100, (PDObs4/PDTotalObs)*100,
(PDObs5/PDTotalObs)*100, (PDObs6/PDTotalObs)*100, (PDObs7/PDTotalObs)*100,
(PDObs8/PDTotalObs)*100, (PDObs9/PDTotalObs)*100]]
EPdata=[EPfinal_digit,EPdigit_decimal,EPSR]
# Write to entire row starting from cell A59
worksheetGraphs.write_row('A1', headings)
# Write to entire column starting from cell A2, repeat for B2 and C2 using index 0,1, & 2
worksheetGraphs.write_column('A2', EPdata[0])
worksheetGraphs.write_column('B2', EPdata[1],percent)
#==================
# Create a Bar Chart
#==================
# Itentify the type of graph chart1 will be
#chart1 = workbook.add_chart({'type': 'bar'})
chart1 = workbook.add_chart({'type': 'bar'})
chart1.add_series({
    'name':       ['Graphs', 0, 1],
    'categories': ['Graphs', 1, 0, 10, 0],
    'values':     ['Graphs', 1, 1, 10, 1],
}) #'name':['Sheet1Name', 0, 1] or you can use 'name': 'StringNameHere'
# Add a chart title and some axis labels.
chart1.set_title ({'name': 'ccy1ccy2_Final_Digit_Distribution'})
chart1.set_x_axis({'name': 'Percbetage_Distribution'})
chart1.set_y_axis({'name': 'Final_Digits'})
#==========================================================
# Set an Excel chart style from 1 to 48 as defined in Excel
#==========================================================
chart1.set_style(11)
# Insert the chart into the worksheet (with an offset).
worksheetGraphs.insert_chart('D2', chart1, {'x_offset': 25, 'y_offset': 10})
#=====================
# Create a column Chart
#=====================
# Itentify the type of graph chart1 will be
chart2 = workbook.add_chart({'type': 'column'})
chart2.add_series({
    'name':       ['Graphs', 0, 1],
    'categories': ['Graphs', 1, 0, 10, 0],
    'values':     ['Graphs', 1, 1, 10, 1],
}) #'name':['Sheet1Name', 0, 1] or you can use 'name': 'StringNameHere'
# Add a chart title and some axis labels.
chart2.set_title ({'name': 'ccy1ccy2_Final_Digit_Distribution'})
chart2.set_x_axis({'name': 'Final_Digit'})
chart2.set_y_axis({'name': 'Percbetage_Distributions'})
#==========================================================
# Set an Excel chart style from 1 to 48 as defined in Excel
#==========================================================
chart2.set_style(11)
# Insert the chart into the worksheet (with an offset).
worksheetGraphs.insert_chart('M2', chart2, {'x_offset': 25, 'y_offset': 10})
#==========================================================================
workbook.close()
```

```
print('Results and Graphs writen to Collusion Tests Results To Be Deleted.xlsx')
```

Glossary & Abbreviations

25-delta: Options delta value rises as options are increasingly In The Money (ITM) and reduces as the options move progressively Out Of The Money (OTM). At The Money Options have value of 50-delta, suggesting a 50% chance of either ending up In The Money or Out Of The Money. A 25-delta is out of the money with only a 25% chance of ending in the money.

Arbitrage: Trading strategy based in the purchase of a commodity, including foreign exchange, in one market at one price while simultaneously selling it in another market at the more advantageous price, in order to obtain a risk-free profit on the price differential, *i.e.* a trading strategy that takes advantage of two or more securities being mispriced relative to each other.

At-The-Money (ATM): An option whose exercise price is the same as the spot price of the underlying currency.

Autoregressive Conditional Heteroscedasticity (ARCH): Time series model for volatilities.

Bank for International Settlements (BIS): Every three years, in September, the BIS issues the Triennial Central Bank Survey, which reports metrics in the foreign exchange market gathered during that year.

Base Currency: The base or unit currency is one unit of $ccy1$ in a $ccy1ccy2$ quoting convention.

Bernoulli: A univariate discrete distribution with only two discrete scenarios, one scenario with value zero (failure) and the other with value 1 (success).

Bid-Ask Spread: The amount by which the ask price exceeds the bid price for an asset in the market. The bid-ask spread is essentially the difference between the highest price that a buyer is willing to pay for an asset and the lowest price that a seller is willing to accept to sell it.

Birth and Death Process: A special case of continuous-time Markov process where the state transitions are of only two types: "births", which increase the state variable by one and "deaths", which decrease the state by one.

Bretton Woods: An agreement negotiated at a 1944 international conference and in effect from 1945 to 1971 that established the international monetary system.

Brownian Motion: The motion of a particle that is subject to a large number of small shocks. See Wiener Process.

BSM: Black-Scholes-Merton Model for pricing European options in stocks, developed by Fisher Black, Myron Scholes and Robert Merton.

Call Option: An option to buy an asset at a certain price by a certain date.

Capital Asset Pricing Model (CAPM): Theoretical model that relates the return on an asset to its risk, where risk is the contribution of the asset to the volatility of a portfolio. Risk and return are presumed to be determined in competitive and efficient financial markets.

Cauchy Distribution: A defined distribution that does not have a mean, variance or higher moments defined. The random variable does not possess a moment generating function.

Central Limit Theorem: states that the sampling distribution of the sample means approaches a normal distribution as the sample size tends to infinity, no matter what the shape of the population distribution.

Chartists: Individuals who uses charts or graphs of an assets historical prices or levels to forecast its future trends. A chartist looks for well-known patterns such as head-and-shoulders or support and resistance levels in securities so as to trade them more profitably.

Cobb-Douglas The functional form of the production function, used to represent the technological relationship between the amounts of two or more inputs and the amount of output that can be produced by those inputs.

Compound Poisson Process: A continuous-time stochastic process with jumps. The jumps arrive randomly according to a Poisson process and the random size of the jumps is specified by a probability distribution.

Continuous Time Process: Process for which the index variable takes values in a continuous range.

CRSP: Center for Research in Security Prices. Provides access to historical data (back to 1926) on US Stock and Indices. It contains day-end and month-end prices on all listed NYSE, Amex, and NASDAQ common stocks along with basic market indices.

Cumulants: A sequence of numbers that describes the probability distribution in a useful, compact way. The first cumulant is the mean, the second the variance, and the third cumulant is the skewness or third central moment.

Decomposition Model: Decomposition procedures are used in time series to describe the trend and seasonal factors in a time series by deconstructing a time series into several components, each representing one of the underlying categories, namely the rates of change or predictability.

Diffusion Process: A solution to a stochastic differential equation with continuous sample paths. A model where the value of the asset changes continuously, with no jumps.

Dominated Security: Security A is dominant over security B, if on some known date in the future, the return on A will exceed the return on B for some possible states of the world, and will be at least as large as on B, in all possible states of the world.

Drift Rate: The average increase per unit of time in a stochastic variable.

European Option: An option that can be exercised only on the day of which it expires.

Fractile: The cut off point for a certain fraction of a sample. If the distribution is known, then the fractile is just the cut-off point where the distribution reaches a certain probability.

Gamma Distribution: A two-parameter family of continuous probability distributions defined in terms of its gamma function.

Gamma Function: An extension of the factorial function to real and complex numbers.

Gamma Variate: A class of random variables that produces a gamma function.

Gaussian Distribution: *See Normal Distribution.*

GBM: A geometric Brownian motion is a continuous-time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian motion with drift.

Generalised Autoregressive Conditional Heteroscedasticity (GARCH): A specification of a dynamic model for volatility. A model for forecasting the volatility where the variance rate follows a mean-reverting process.

Geometric Brownian Motion: Stochastic process assumed for asset prices where the logarithm of the underlying variable follows a generalised Wiener process.

GK: Garman and Kohlhagen modified Black-Scholes-Merton model for pricing European foreign exchange options. Developed by Mark Garman and Steven Kohlhagen.

Heteroscedasticity: The variance of a series is not constant throughout the sample.

Impulse Function: A function that is zero everywhere but at the origin where it is infinitely high.

In-The-Money (ITM): Circumstance in which an option is profitable, excluding the cost of the premium, if exercised immediately.

Indicator Function: A function defined on a set X that indicates membership of an element in a subset A of X having a value 1 for all elements of A and 0 for all elements of X not in A.

Interest Rate Parity: Theory that the differences in national interest rates for securities of similar risk and maturity should be equal to but opposite in sign to the forward exchange rate discount or premium for the foreign currency.

International Monetary Fund (IMF): An international organisation created in 1944 to promote exchange rate stability and provide temporary financing for countries experiencing balance of payments difficulties.

Itô's Process: A stochastic process where the change inthe variable during each short period of time of length Δt has a nomral distibution. The mean and variance of the distibution are proportional to Δt and are not necessarily constant.

Kurtosis: A measure of the fatness of the tails of a probability distribution of a real-valued random variable.

Leptokurtic: A series which has a higher peak at the mean and fatter-tails than a normal distribution with the same mean and variance.

LIBOR: London interbank offered rate. The rate bid by banks on Eurocurrency deposits, *i.e.* the rate at which a bank is willing to lend to other banks.

Limit Order: Direction given to a broker to buy or sell a security or commodity at a specified price or better.

Market Order: Direction given to a broker to buy or sell a security or commodity at the best available current market price.

Markov Process: Stochastic process where the behaviour of the variable over a short period of time depends solely on the value of the variable at the beginning of the period, not on the past history.

Mixture Distribution: The probability distribution of a random variable that is derived from a collection of other random variables.

Multinomial Prior: The number of distinct way to permute a multiset of n elements with k_i multiplicities of each of the distinct elements before some evidence is taken into account.

Noise Traders: A term used to describe investors who make decisions regarding buy and sell trades without the support of advanced fundamental analysis.

Non-Linearities: A system in which the change of the output is not proportional to the change of the input.

Order Flow: The net of buyer-initiated and seller-initiated orders; it is a measure of net buying pressure.

Out-of-The-Money (OTM): An option that would not be profitable, excluding the cost of the premium, if exercised immediately.

Outliers: Data points that do not fit in with the pattern of the other observations and that are a long way from the fitted model.

Overshooting: Behaviour in financial markets in which a major market adjustment in price changes "overshoots" or surpasses the likely value it will settle at after a longer adjustment period, akin to a market "overreaction".

Paretian Distribution: A power law functional relationship between two quantities where a relative change in one quantity results in a proportional relative change in the other quantity, known as the 80/20 rule.

Partial Differential Equations: (PDE) is an equation which imposes relations between the various partial derivatives of a multivariable function.

Pip: An acronym for "percentage in point" and the smallest price move that an exchange rate can make based on market convention.

Platykurtic: Describes a statistical distribution with thinner tails than a normal distribution. Because this distribution has thin tails, it has fewer outliers.

Poisson Distribution: A statistical distribution that shows how many times an event is likely to occur within a specified period of time. Poisson is a discrete function with events being measured as occurring or not occurring only.

Poisson Process: Process describing a situation where events happen at random. The probability of an event in time Δt is $\lambda \Delta t$, where λ is the intensity of the random process.

Power-Law: In statistics, a power law is a functional relationship between two quantities, where a relative change in one quantity results in a proportional relative change in the other quantity, independent of the initial size of those quantities: one quantity varies as a power of another.

Price Attraction Hypothesis: Quoted Prices are subject to rounding human error.

Price Clustering: Price clustering refers to the fact that not all the available digits are used equitably.

Price Discreteness: In the FX market exchange rates are truncated to a fixed number of digits, this is referred to as price discreteness.

Price Resolution Hypothesis: Quoted prices are a compromise between increased accuracy and ever longer prices.

Probability Density Function (p.d.f.): Relationship or mapping that describes how likely it is that a random variable will take on a value within a given range. That is the value of any point in the sample space can be interpreted as providing a relative likelihood that the value of a random variable would equal that point.

Purchasing Power Parity (PPP): Theory that the price of internationally traded commodities should be the same in every country, and hence the exchange rate between the two countries should be the ratio of prices in the two countries.

Put Option: An option to sell an asset for a certain price by a certain date.

Quote Currency: The quote currency is number of units of $ccy2$ in a $ccy1ccy2$ quoting convention required to purchase one unit of $ccy1$.

Random Process: A time varying function that assigns the real valued outcome of an experiment to each instant of time.

Random Walk: Simple model where the current value of a series is simply the previous value acted on by a white noise (error) term. Therefore the optimal forecast for a variable that follows a random walk is the most recently observed value of that series.

Rational Expectations: A concept and modeling technique that is used widely in macroeconomics. The theory posits that individuals base their decisions on three primary factors: their human rationality, the information available to them, and their past experiences. The theory defines this kind of expectations as being the best guess of the future.

Reporting Dealers: Dealers executing trades and managing risk on their accounts and on behalf of their customers.

Risk Reversals: An option position that consists of being short (selling) an out-of-the-money put and simultaneously long (buying) an out-of-the-money call, both with the same maturity.

ECN: Electronic Communication Network. A computerised system that automatically matches buy and sell orders for securities in the market.

ISO: Currency short codes from the International Organisation for Standardisation.

NASDAQ: National Association of Securities Dealers Automated Quotations, a global electronic marketplace for buying and selling securities, as well as the benchmark index for U.S. technology stocks. Nasdaq was created by the National Association of Securities Dealers (NASD) to enable investors to trade securities on a computerized, speedy and transparent system.

NYSE: New York Stock Exchange.

Sample Space: The set of all possible outcomes.

Scaling Factor: Multiplication factor with the purpose of bringing the pip to the left of the decimal point.

Skewed: Frequency distributions which are not symmetric.

Skewness: Standardised third moment of a distribution that shows whether it is symmetrical around its mean value.

Stable Distribution: A distribution is stable if a linear combination of two independent random variables has the same distribution as the individual variables upto location and scale parameters. The stable distribution is an application of the Generalized Central Limit Theorem, which states that the limit of normalized sums of independent identically distributed variables is stable.

Stationary Process: A stochastic process whose unconditional joint probability distribution does not change when shifted in time. Consequently, parameters such as mean and variance also do not change over time.

Stationary: A time series whose statistical properties such as mean, variance and autocorrelation are all constant over time.

Stochastic Process: Process describing the probabilistic behaviour of a stochastic variable.

Technical Analysts: Noise traders whose trading strategies are usually unrelated to company fundamentals.

Tick: A measure of the minimum upward or downward movement in the price of a security.

Volatility Smile: The variation of implied volatility with strike price.

White Noise: A Process with a fixed mean and variance but no other structure (zero autocorrelation for all lags). The error term in a regression model is usually assumed to be white noise.

Wiener Process: A stochastic process where the change in a variable during each short period of time length Δ_t has a normal distribution with a mean equal to zero and a variance equal to Δ_t.

Notations

Av Average or mean value.

B^{-1} Positive definite weighting matrix.

$B_i(t)$ Buy price.

CP_t Closing exchange rate.

D^p N-dimensional vector.

E Expected return.

$E[R]$ Expected value of random variable R.

F_t Vector of fundamental variables.

F_x The first partial derivative.

$Hi(\phi_i')$ Highest percentage frequency at digit i.

Ku The moment coefficient of the sample kurtosis.

$Lo(\phi_i')$ Lowest percentage frequency at digit i.

MP_t Mid point quote from the prevailing bid and ask quotes before a transaction.

Max Maximum value.

Min Minimum value.

$N(t)$ Poisson process with parameter λt.

N_d Notional units of domestic currency.

N_f Notional units of foreign currency.

$P(A|B)$ Conditional probability of A given B.

Q Percentage at each final digit i, if no clustering occurs.

$R(t)$ Return over unit period of time.

R_t Dealer return.

S Foreign exchange spot rate.

SEK Standard Error of Kurtosis.

SES Standard Error of Skewness.

SR Standardised Range.

S_d Bid-Ask Spread.

S_t Foreign exchange spot rate at time t.

$S_i(t)$ Foreign exchange spot rate i at time t.

S^e_{t+1} One period ahead expected foreign exchange spot rate given information available at time t.

Sk The moment coefficient of the sample skewness.

T A time step which controls the time scale of the model.

TP_t Transaction price.

U_{it} Independent and identically normally distributed random variable.

V_t Unobservable fundamental value of stock in the absence of transactions costs at time t.

V^*_t Interim value of V_t due to a buy-sell imbalance.

$Var[R]$ Variance of random variable R.

$Z(t)$ Denotes the natural logarithm of the price of a security at time t.

Z_{Ku} Sample kurtosis test statistic.

Z_{Sk} Sample skewness test statistic.

ΔP_t Change in Price.

$\Delta S_i(t)$ Change in exchange rate.

Δt Change in the time interval t.

Δz Change in the variable z.

$\Gamma(x)$ Gamma function for x.

α Characteristic exponent which measures the degree of peakedness and the fatness of the tails of a stable distribution. Also known as the stability parameter.

α^* Private information component.

$\bar{B}_i(t)$ Sell price.

$\bar{\sigma}$ Average standard deviation.

$\bar{\nu}'$ Average logarithm rate product.

β Temporary buy-sell imbalance component.

β' Vector of factor loadings.

χ^2 Chi-squared distribution.

$\epsilon \sim (0,1)$ is a standardised normal random variable.

γ Gamma distribution shape parameter.

$\gamma*$ Scale factor for a symmetrical Lévy stable distribution.

γ_1 Skewness of compound events distribution.

$\hat{\alpha}$ An estimate of the characteristic exponent α.

$\hat{\hat{\alpha}}$ An average of the estimates $\hat{\alpha}$ of α.

λ Parameter of the intensity of the Poisson process, that is the average number of events per interval.

λ_p The difference between a dealers buying and selling price.

λ_{dm} The proportion of observations associated with an information set.

λt The number of random events occurring in time t.

ln Natural logarithm of a number, its logarithm to the base of the mathematical constant e.

μ Mean of a random variable.

ν Gamma process variance rate.

ν' Logarithm rate product.

ϕ'_i Percentage of final digit observations, at i, as a proportion of the total population.

π Geometrical value Pi.

$\pi^+\pi^-$ Conditional probabilities.

$\prod$ Product - product of all values in range of series.

σ Standard deviation of a random variable.

σ^2 Variance of a random variable.

$\sum$ Summation - sum of all values in range of series.

θ_p Pareto exponent.

θ_{ce} Mean of a sequence of normally distributed, mutually independent random variables.

θ_{dm} Parameter vector.

υ Rate product.

ε_t Public information shock.

$\phi_x(u)$ Characteristic function for the probability distribution of X.

a Positive constant.

$a_i'(t)$ The ith dealers price movement at time t.

c Gamma distribution scale parameter.

c' Constant specifying the dealers response to changes in the exchange rate.

c_p Distribution shift parameter.

d_p Distribution scale factor.

e_t Error term.

$f(x)$ Probability density function of random variable X.

f_i Denotes independent fluctuations.

g Represents an interaction function.

h^p Positive constant.

ip Relative industrial production.

ir Nominal interest differential.

m Ratio of domestic to foreign nominal money supply.

of_t Order flow.

p Inflation differential.

q Dividend yield.

r_t Stock return observation.

tb Relative cumulated trade balances.

x^p Column vector of variables.

I_i Homogeneous information sets.

Y_k Mutually independent random variable with mean θ_{ce} and variance σ_2^2.

$l(A|B)$ Maximum likelihood of A given B.

$n_{i(j)}$ Number of observations at final digit $i(j)$.

$||$ Absolute value.

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