DE FINETTI AND SAVAGE ON THE NORMATIVE RELEVANCE OF IMPRECISE REASONING: A REPLY TO ARTHMAR AND BRADY

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This paper examines the claim that de Finetti and Savage completely rejected the notion of indeterminate, as distinct from imprecise, probabilities. It argues that their examination of imprecise reasoning refers both to descriptive and normative issues, and that the inability for a decision-maker to commit to a single prior cannot be limited to measurement problems, as argued by Arthmar and Brady in a recent contribution to this *Journal*. The paper shows that de Finetti and Savage admitted that having an interval of initial probabilities may sometimes have normative relevance, thereby leaving an opening for indeterminate probabilities.

1. INTRODUCTION

Bruno de Finetti is famous for the uncompromising purism with which he advocated subjectivist-Bayesianism, for being a «Radical Probabilist» *par excellence* as he was affectionately remembered at the 2006 conference marking the centenary of his birth (Galavotti 2009). This reputation stems from his strong denial of any role for objective elements in both the theory and the practice of statistics (Jeffrey 1989). De Finetti's position in this respect, already evident in his very early work (de Finetti 1930, 1931) was unwavering over the length of his career: more than forty years after starting out he would open his magnum opus, *Theory of Probability*, with the iconic phrase «Probability does not exist» (de Finetti 1974, x).

De Finetti's operational definition of probability reflects the same uncompromising spirit. In his theory, probability is defined in terms of betting quotients, derived as the ratio between the sum of money an individual would be willing to bet on the occurrence of a certain event in exchange for a given prize, and the prize itself. Under the assumption that the individual is obliged to bet either on the event or its opposite, a sharp numerical probability can be derived for any event (de Finetti 1937). Indeed, *all* subjective probabilities become numerically definite by definition from a strictly subjectivist viewpoint. Moreover, this notion of probability is intended as an instrument for making choices under uncertainty, applicable to choices related to any kind of event, from the spin of a

roulette wheel to the outcome of a horse race. *All* uncertainties become measurable, on this view, thereby obviating the need to distinguish between risky and uncertain situations, regardless of claims to the contrary by economists such as Keynes (1921) and Knight (1921). The clarity and forcefulness with which de Finetti stated this position did much to make it the basis of the new Bayesian mainstream, as represented by Leonard Savage's (1972 [1954]) axiomatic system.

Viewed in this light, the possibility that there may be room for imprecision in probabilistic reasoning after all in de Finetti's theoretical construct, and in particular in its development and defence pursued jointly with Savage, is therefore an intriguing one. We examined this possibility in Feduzi *et al.* (2012) and showed that there is evidence in de Finetti's work that suggests that he was more receptive to a distinction between risk and uncertainty than is usually admitted. To be sure, he rejected Knight's sharp distinction between probabilistically measurable risk and unmeasurable uncertainty. But his 1967 study of the application of the economics of uncertainty to insurance, yet to be translated from the original Italian, suggests a theoretical case for uncertainty even when individuals are endowed with sharp subjective probabilities. Indeed, de Finetti (1967a, 36-38) conceded that Knight's distinction may become relevant when different individuals «sensibly disagree» on the probability of the occurrence of an event, with «risks» corresponding to cases in which one finds only minor discrepancies in valuations made by different individuals—which is «what renders them insurable».[1] He was thus led to suggest that a situation of uncertainty might be conceived as one in which individuals' opinions about the probability of the occurrence of a given event «sensibly differ», and which, contrary to the predictions of the standard model of risk exchange inspired by the subjectivist treatment of uncertainty (Borch 1967), may lead insurance markets to fail even where individuals are able to attach sharp numerical probabilities to the contingencies concerned.[2]

The question that then arises is what this admission says about de Finetti's views on the foundations of probability. Is there any evidence to suggest that he might have been less stringent a Bayesian with respect to specific cases coloured by ambiguity? Might the differences of opinion admitted with respect to a group of individuals represent the epistemic state of a single individual,

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[1] Quotations from de Finetti's works unavailable in English are translated from the original Italian by the authors, including passages attributed to Savage in DE FINETTI AND SAVAGE (1962).

[2] DE FINETTI (1967a, 37) suggests that the degree of difference between individuals' subjective probabilities depends on the particular circumstances under which those judgements are elicited. He is therefore sceptical of attempts to draw a clear line between cases in which there is perfect uniformity of judgements and cases in which there is not, and rejects Knight's terminology on the grounds that it may give the impression that the boundary between risk and uncertainty is clear-cut rather than fuzzy.

possibly in very special instances, but be of theoretical interest? Our claim in Feduzi *et al.* (2014) is that de Finetti's largely unnoticed positive attitude towards instances of uncertainty indicates a more favourable view of imprecise reasoning than is usually recognised. To show this, we concentrated mainly on his collaboration with Savage, which occurred at a stage when even some of their subjectivist colleagues were voicing reservations about the subjectivist view in decision theory (Ellsberg 1961, Fellner 1961) and statistics (Smith 1961). Focusing mostly on de Finetti and Savage's assessment of Cedric Smith's foundational paper of the statistical approach to approximate reasoning (Walley 1991), we attempted to show that the way de Finetti defended his position against Smith's demonstration that interval-valued probabilities may be derived from choice behaviour—Smith following de Finetti himself in the adoption of an operational perspective—amount to more than a mere acceptance of the existence of descriptive violations of what is intended as a normative standard. That is to say, rather than interpreting imprecision as nothing more than an empirical violation of Bayesian rationality, one that does not disturb—indeed justifies—the theory as a normative tool, de Finetti and Savage admit that imprecise reasoning may sometimes be acceptable even from a normative perspective.

Our assessment of the subtleties of de Finetti's understanding of uncertainty was recently criticized in this *Journal* by Arthmar and Brady, who claim that we overlook the important distinction between imprecise and indeterminate probabilities. Since we disregarded this difference, according to Arthmar and Brady (2016, 107), we failed to see an important difference between de Finetti and Savage «who completely discard the concept of indeterminate probabilities», and Keynes and Knight who reserved a place for indeterminacy as well as imprecision in probability and decision-making. The difference between imprecise and indeterminate probabilities may be described as follows. Following Levi (1985, 392), a probabilistic assessment does not conform to a «strict Bayesian viewpoint» when in an otherwise subjective framework, «rational agents often do not and should not regard exactly one real-valued probability function to be permissible for use in assessing expected utilities». Strict Bayesians allow that «human agents … ought not to be expected to be able to identify their strictly Bayesian credal probability judgements with full numerical precision», but regard this as an issue of *imprecise* probability priors stemming from measurement problems. Measurement problems may be relevant in practice for an actual decision-maker who cannot see with clarity the details of a decision problem, but do not bother a rational decision-maker who cannot suffer permanent lack of information. In contrast, *indeterminate* probability priors arise when it is «rational to make no determinate probability judgement» and where a refusal to do so may derive from «a very clear and cool judgment that on the basis of the available evidence, making a numerically determinate judgment would be unwarranted and arbitrary» (Levi 1985, 396).

In spite of their differences in other respects, writers such as Koopman (1940), Good (1952), Smith (1961), all explore the possibility of using sets of probability priors or upper and lower probability functions for a systematic characterization of degrees of belief. More generally, and whether they were conscious of this or not, authors using a less strict Bayesian approach were working in the footstep of Keynes (1921), who associated uncertainty with indeterminate, as opposed to imprecise, probability judgements, and Knight (1921), who saw uncertainty as something not representable through probabilities. In contrast, de Finetti and Savage, were committed to characterizing degrees of belief by way of single probability priors, in a subjectivist set-up that, as we have already noted, was intended to transform uncertainty into numerically definite subjective probabilities. Thus, Arthmar and Brady argue, any concession by de Finetti and Savage with respect to a decision-maker not being prepared to commit to a single prior must be interpreted as admitting imprecision, something a rational decision maker would eventually rectify via updating, rather than indeterminacy.

Our aim in this paper is to respond to Arthmar and Brady by arguing that they fail to appreciate the importance in our account of the difference between the descriptive and the normative in probabilistic reasoning, and how this difference relates to peoples' position on imprecise versus indeterminate probability. The claim that de Finetti and Savage completely rejected the notion of indeterminate probabilities can be dismissed on the basis of the normative value they attributed to Smith's considerations, in our view, regardless of an explicit account of the distinction between imprecise and indeterminate probabilities.[3]

2. De Finetti and Savage's reaction to Smith

As author of the first comprehensive presentation of what came to be known as the Bayesian approach to decision-making and statistics, Savage became widely regarded as its champion. But while Savage's approach was enthusiastically endorsed by decision theorists (Luce and Raiffa 1957, Arrow 1958), the response from statisticians was not encouraging (Wallis 1981). For instance, the discussion following Savage's presentation of the Bayesian viewpoint at the Joint Statistics Seminar at Birkbeck and Imperial Colleges in London in 1959 reflects the fierce opposition to his ideas among a vast majority of frequentist statisticians (Savage 1962a). Savage nevertheless found a major ally in de Finetti, in his quest to promote a paradigmatic shift in statistics, who he recognized

[3] Vicig and Seidenfeld (2012) anticipate Arthmar and Brady's views in many respects, and note that fundamental contributions made by de Finetti—especially coherence of subjective probabilities and exchangeability of random variables—form part of an active imprecise probability research agenda in statistics. But they maintain that he personally ignored the possibilities of using it to analyse uncertainty.

as a major source of inspiration (Savage 1972 [1954], 4). De Finetti and Savage stayed in close and constant contact in those years, and their agreement on almost all issues was so profound that it is difficult to find notable differences between the two when they corresponded about their critics. In the early 1960s, then, the historical background was one that saw de Finetti and Savage focused on showing how the subjective probability approach could foster the spread of new statistical tools and operational methods for inference, all in a hostile environment in the statistical arena (Fienberg 2006).

However, the subjectivist approach was also suffering friendly fire at the time. Ellsberg (1961) and Fellner (1961), both declared subjectivists, questioned Savage's probability approach, on the grounds that it was not rich enough to deal with many instances of actual decision environments, and that the way it allowed decision-making under uncertainty to be subsumed into the framework von Neumann and Morgenstern (1947) had devised for risk was unwarranted. Moreover, one of the very few English statisticians committed to the subjectivist approach, Cedric Smith, advanced a similar criticism. Smith presented the need to fix initial probabilities as an obvious precondition for the application of Bayesian methodology, but investigated the possibility that personal probabilities are not necessarily sharp, presenting his work as a generalization of Savage's subjective approach that admitted imprecision. Smith measured «imprecise» beliefs by means of betting quotients arguing that to be prepared to bet on an event at a certain maximum price does not equate to being prepared to bet against it at an infinitesimally higher one. Personal betting quotients could then be interpreted as upper and lower probabilities and the person be attributed an interval of initial probabilities. The fact that he adopted an operational perspective *a la* de Finetti made it necessary for de Finetti and Savage to comment on it.[4]

As noted in Feduzi *et al*. (2014), de Finetti's position on interval-valued probabilities is best reflected in his 1962 joint paper with Savage called «Sul modo di scegliere le probabilità iniziali» («How to choose the initial probabilities»). This paper, which presents the subjectivist justification for the use of initial probabilities and clarifies the meaning of fixing such probabilities arbitrarily, is notable because it includes an entire section on the relevance of Smith's contribution to the development of Bayesian statistics. An introductory note explains that this section was added in September 1961, while the main part had already being drafted following a sabbatical Savage spent

[4] SMITH (1965, 478) illustrated his viewpoint as follows: «if I am willing to bet 2 to 1 on sun against rain, and 1 to 4 on rain against sun, this means that I regard sun as between 2 and 4 times as probable as rain; and I do not need to be more precise than this». As a result the elicitation of probabilities from choices entails that «probabilities and utilities are no longer uniquely defined, but, in accordance with human vagueness and imprecision, they are only determined within a certain range».

in Rome in 1959. Concentrating on this 1961 addition makes it possible to see how de Finetti and Savage reacted to criticism emanating from within the subjectivist camp.[5]

The paper discusses a number of foundational issues, including whether «inexactly determined» and «fuzzy» initial opinions can be expressed through an exact probability value. It turns out that some tension emerges between the strict subjective interpretation and de Finetti's further elaboration with Savage on this point. On the one hand, the idea that there can be no exact knowledge of initial probabilities is taken to be «meaningless» (de Finetti and Savage 1962, 94) and claims to the effect that exact probabilities can be replaced by probability intervals are described in the English summary as posing «more severe problems that they are intended to resolve» as they entail the identification of precise upper and lower values (Savage 1962b, 150).[6] On the other hand, in a paragraph added in the final draft of the paper, de Finetti and Savage admit that, «it is often practically impossible to anyone to state that … the probability which he can attribute to a certain event has a precise value» (de Finetti and Savage 1962, 95).

This passage alludes to more than measurement problems, however, because it reflects Savage's apparent concern about suggesting a version of Bayesianism that was simply too demanding. Added in the 1961 version of the paper—which reproduces part of the correspondence between the two authors ensuing from the preliminary draft, with each author speaking for himself—is Savage's claim that:

> we seem to argue [in the preliminary draft] that imprecision in probability judgements can be always removed, after providing enough effort … [but] this conclusion is not in harmony with my experience and introspection.

Savage finds it difficult to insist on numerical precision «with respect to many highly relevant and concerning events», for instance, the «probability of a world war in a near future» (de Finetti and Savage, 130). The initial probability may be impossible to make precise in this instance:

> Of course there exist a great many data relevant to this event, that I could gather and arrange, and also important ways in which I could arrange my thoughts on these, but I could not hope nonetheless that the result of my efforts could entail a probability.

And he (de Finetti and Savage 1962, 131) concludes:

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[5] The paper has never been translated in English but is known in the statistical literature courtesy of a long English summary by SAVAGE (1962b). Savage's English summary, though, appears to have been written after the first draft was completed, and does not include reference to the additional section on imprecision. The English summary is therefore missing a crucial part of the story.

[6] Later, DE FINETTI (1974, 334) made his point as follows: «That an evaluation of probability often appears to us more or less vague cannot be denied; it seems even more imprecise, however (as well as devoid of any real meaning), to specify the limits of this uncertainty».

Specifically, I maintain that our current effort is devoted to contrast some of the difficulties related to the phenomenon of vagueness and imprecision, and, though we can make something to overcome this difficulty, I do not think we can pretend to eliminate it completely.[7]

De Finetti seems to be the more cautious of the two authors, but he (de Finetti and Savage 1962, 131) agrees that:

we do not really claim that we can seriously attribute a precise value to every probability … but only that it can often be done with adequate approximation ... and most of all that if this is not enough it cannot be for any other decision method as well.

The conclusion thus seems to be that the descriptive issue is more relevant than previously admitted: even though imprecision «is not something on which one can meaningfully theorise», it can constitute an «actual epistemic state» of the individual facing uncertainty, whose nature is «difficult to be made precise in a convincing manner» (de Finetti and Savage 1962, 133-134).

It should be remembered that, when introducing his axiomatic structure for decision-making under uncertainty, Savage (1972 [1954], 16) distinguished a «small» world, in which he considered his theory genuinely valid, and a «grand» world, in which not all the pay-off relevant events can be enumerated beforehand so that one should better «cross that bridge when you come to it». Before concentrating on small worlds, he rejected the idea that all worlds can be treated as small as «ridiculous». As noted by Binmore (2009), Savage's small/grand distinction delineates a field of application of his theory that resembles the Knightian distinction between risk and uncertainty. Moreover, Savage's reaction to the Ellsberg Paradox was not as direct as that to the Allais Paradox, when he reacted to his failure to obey his own axioms in Allais's test by claiming that, on reflection, he would have reversed his preferences (Savage 1972 [1954], 103). Savage was reported to be among deliberate violators of his own axioms when confronting Ellsberg's urns (Ellsberg 1961, 656), but there is no evidence in any of his publications that he changed his mind about the significance of his violation. And he did not appear to object to Ellsberg's and Fellner's argument about the normative relevance of vagueness in probability judgements, commenting only that: «some have tried to reflect the phenomenon of vagueness within the theory, while others believe

[7] Savage here seems to be referring to updating that does not lead to sharp probability judgements. In their comment on our paper, Arthmar and Brady take a different view and claim instead that de Finetti and Savage regarded any indecision in attributing an initial probability as being limited to the very initial stage of Bayesian analysis, to be solved by updating. They also claim that this is not so in Keynes's probability theory. These are larger issues than cannot be dealt with here, not least because we find difficult to make general claims about updating in Keynes's theory. But even leaving aside the noted subtleties in Savage's position, it is worth remembering that there is no evidence that Keynes was critical of Bayes in his *Treatise on Probability* and that de Finetti's only detailed comment on Keynes's *Treatise* is highly appreciative of Keynes's theory of induction (DE FINETTI 1985 [1938], 84-85).

that, though vagueness must somehow be reckoned with, its nature defies formalization» (Savage 1967, 308). Savage's claim from the 1962 paper reported above, then, seems difficult to reconcile with the idea that he and de Finetti followed Ramsey in viewing «uncertainty as being *strictly* a measurement error problem», as claimed by Arthmar and Brady (2016, 110, emphasis added).

What is more, in its treatment of Smith's contribution, the 1962 paper signals de Finetti and Savage's normative doubts about the assumption of sharp priors (Feduzi et al. 2014, 16-18). As just noted, Smith worked with interval-valued probabilities, but, differently from Keynes, Koopman and Good, did not interpret probabilities as intuitive judgements. Rather, he adopted an operational perspective. Also he showed how the fundamental principles of avoiding sure loss and coherence could be applied to the axiomatic context of interval-valued probabilities (Walley 1991). In the additional sections devoted to Smith's «particularly elaborate analysis» of Koopman's and Good's idea «to make imprecision precise», de Finetti and Savage (1962, 133) admit that Smith's approach provides an operational criterion to precisely determine the two limiting lower and upper probability values: «in Smith's case», they concede, «… the objections about the precision of the extreme values do not hold» (de Finetti and Savage 1962, 135). This concession contradicts what they claimed before and, significantly, what the English summary reports. But de Finetti and Savage also make a specific concession with regard to the normative justification of imprecise reasoning, presenting an important case in which they accept Smith's argument. Smith's considerations are said to «express what can be said of a certain behaviour when one has an incomplete knowledge of the opinions justifying a decision». Sounding as if they are trying to make up their own mind, de Finetti and Savage (de Finetti and Savage 1962, 141-142) go on: while «an opinion implying an indeterminate probability concerning a certain event is not admissible», it may be possible:

> to know imperfectly an opinion, and thus to be capable of identifying only partially the preferences which the opinion implies (in a complete manner) among alternative possible decisions.

The implication is that Smith's approach may be of help in cases in which one has «partial knowledge of a preference» so that the probability of an event can be said to be indeterminate. While de Finetti and Savage did not offer generalizations of philosophical interest on this issue, they (1962, 142) do provide an example in order to illustrate their point, that is, the case of a single individual who experiences a «kind of personality dissociation … as indicated also by Smith». They see this case as representing the situation of an individual who has «various souls leaning towards contrasting opinions», possibly because they are in doubt about whether to rely on their own sharp prior or on ones they have been made aware by consulting experts they respect. De Finetti and Savage admit that what is obvious for group decision-making even in a strictly Bayesian set-up—

with the sharp probability priors of each individual generally diverging one from the other (de Finetti 1955)—could be ascribed to a single individual who, due to conflicting evidence, such as diverging experts' opinions, is endowed with a set of probability priors.

Situations of conflicting evidence are of course quite common in practical decision situations, but what matters for our assessment is that it is not clear that admitting their relevance for decision-making should be considered a failure of rationality. For sure, it amounts to a failure of strict Bayesian rationality, but not of the Bayesian viewpoint in general and of Keynes's view in the *Treatise on Probability*.[8] Indeed, this was the point made by Ellsberg in his critique of Savage. Even simple urn examples, not necessarily complex real life situations such as the uncertainty concerning «a world war in the near future», showed that many individuals fail to conform to strict Bayesian rationality even after thorough consideration of Savage's axioms, and that this might be a problem with the normative strictures of the theory rather than any shortcomings in the reasoning of the individuals concerned (Ellsberg 1961, 660-661). Probabilities are indeterminate, in such instances as a result of what Levi above calls «a very clear and cool judgement». It is also worth noting that when distinguishing between imprecise probabilities generated either by indeterminacy of beliefs or incompleteness due to difficulties in assessment, Peter Walley (1991, 214), possibly the major contributor to the development of imprecise reasoning in statistics, lists conflict between expert opinions as a notable example of «imprecision [that] reflects unavoidable indeterminacy rather than incomplete modelling».

A crucial part of the evidence for our argument that de Finetti and Savage's concession to Smith had normative significance is the rarely mentioned taxonomy of probability theories created

[8] ARTHMAR AND BRADY (2016, 107) maintain that «what Feduzi *et al*. call Keynes's 'non-numerical' probabilities are actually Keynes's interval-valued probability estimates», and mildly take us to task for seeming «oblivious to the Boolean framework used by Keynes in the TP» (which informed the interval-valued approach laid out in Chapter 15 of that book). However some of us had noted in earlier work the place Keynes reserves for interval-valued probabilities in Chapter 15 of a *Treatise on Probability* (RUNDE 1994a; 1994b, BASILI AND ZAPPIA 2009, ZAPPIA 2015) and, albeit admittedly without making explicit links to Chapter 15 of *A Treatise in Probability* or Boole himself, have all written at length on interval-valued probability in various papers on Keynesian uncertainty (both in the papers Arthmar and Brady reference (FEDUZI *ET AL*. 2012, 2014) and also in RUNDE (2001) and FEDUZI AND RUNDE (2011)). For our part, we agree with Arthmar and Brady that there is considerable continuity between Keynes's earlier views on probability and parts of his later economic writings on the nature and effects of uncertainty, and we see their emphasis on the theme of interval-valued probabilities as an important corrective to the literature on Keynesian uncertainty in which it is generally ignored (BRADY AND ARTHMAR 2012). But we would part ways on their broad characterisation of a *Treatise on Probability* as an «interval-valued theory of probability». We do so because we feel is misleading for his whole theory being characterised in terms of what is an adjunct to it, a method of approximation which, in KEYNES's (1973 [1921], 177, emphasis added) own words, «may *occasionally* be useful».

by de Finetti (1967b) for an entry on the theory of probability in the *International Encyclopaedia of Social Sciences*. As we noted in Feduzi *et al*. (2014), de Finetti divides philosophical interpretations of probability into objective and subjective approaches. Subjective theories are classified under the headings of «psychological», «consistent» and «rational» with respect to the behaviour under uncertainty they intend to examine (de Finetti 1967b, 499-501). While the consistent subjective probability theory branch includes Ramsey, Savage and de Finetti himself, and the rational branch comprises the logical approach of Keynes and Jeffreys, psychological theories are not linked explicitly to any author. De Finetti argues that the psychological subjective theory of probability emerges from then recent experimental studies of the actual behaviour of individuals under uncertainty, but makes it clear that actual behaviour that diverges from coherent behaviour cannot be used to object to normative theories, since normative theories like the subjective and rational variants are intended to state «what behaviour is good or bad», irrespective of the actual behaviour examined in psychological theories (de Finetti 1967b, 500). De Finetti then goes on to acknowledge that some theories do not quite fit his scheme, particularly because of their insistence on probabilities that «may be noncomparable, and hence nonnumerical». He mentions Ellsberg as a variant of subjective psychological theories that permits non-comparability, and which, as a psychological theory, he dismisses as a possible counterexample to a normative theory. Crucially, however, he classifies Smith's interval-valued probabilities approach as a variant of the subjective consistent kind, thereby bringing Smith's representation of non-comparability into the fold of normative theories. By separating Smith from psychological studies, in other words, de Finetti takes a clear step towards acknowledging the issue of indeterminacy and admitting potential normative status for theories allowing for interval-valued probability priors.[9]

3. Conclusion

Contrary to what one would expect from an orthodox proponent of the subjectivist-Bayesian approach, de Finetti's rejection of interval-valued probabilities needs to be qualified. His collaboration with Savage in the 1960s amounts to a joint effort aimed at both clarifying the relevance of their viewpoint for statistical analysis and accounting for the criticism coming from adherents to their subjectivist viewpoint. In their reflection on Smith, de Finetti and Savage refer to

[9] The final draft of the de Finetti's entry had been circulated already in 1963 on the suggestion of Savage, who edited it extensively. Savage was accustomed to translating parts of de Finetti's work from the Italian for circulation among English speaking colleagues. In the 1960s he edited the translation of every work by de Finetti, including the 1964 translation by Kyburg of the famed 1937 foundational paper. In the editing of de Finetti (1967b) there is no objection to the choice to put Smith among consistent probability theories.

both descriptive and normative issues. They clearly acknowledge that the assumption of precise numerical probabilities often fails descriptively, thereby admitting imprecision as noted by Arthmar and Brady in their contribution to this *Journal*. But de Finetti and Savage also suggest that an individual's inability to commit to a single prior may sometimes be acceptable even from a normative point of view, thereby, and contrary to what Arthmar and Brady argue, leaving an opening for indeterminate probabilities.

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